FINAL LIFTING LEMMA Assume X is path connected and locally path connected. Let

$$p : (E, e_0) \longrightarrow (B, b_0)$$

be a covering space and

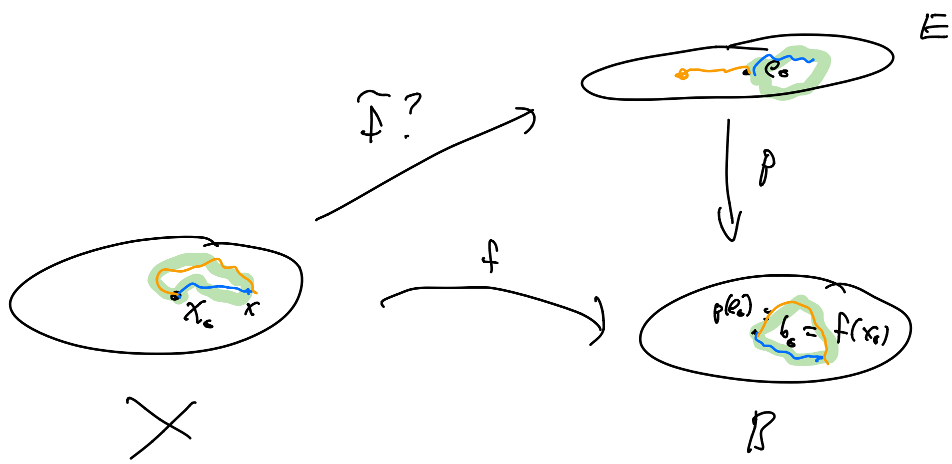
$$f : (X, x_0) \longrightarrow (B, b_0)$$

a map. Then f has a lift

$$\tilde{f} : (X, x_0) \longrightarrow (E, e_0)$$

if and only if $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$.

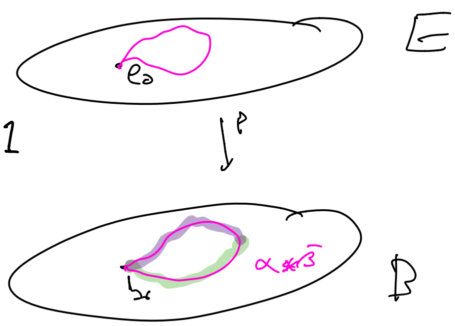
If the lift exists it is unique.



LEMMA Let $\alpha, \beta : [0,1] \to B$ be paths with $\alpha(0) = \beta(0) = x_0$ & $\alpha(1) = \beta(1)$.
Let $\tilde{\alpha}, \tilde{\beta} : [0,1] \to E$ be the lifts with $\tilde{\alpha}(0) = \tilde{\beta}(0) = e_0$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$ if and only if $[\alpha * \bar{\beta}] \in p_*(\pi_1(E, e_0))$.

$\in \pi_1(B, b_0)$

$$\alpha * \bar{\beta}(t) = \begin{cases} \alpha(2t) & 0 \le t \le 1/2 \\ \beta(2-2t) & \frac{1}{2} \le t \le 1 \end{cases}$$



Given $\widetilde{\alpha * \bar{\beta}}$ write $\tilde{\alpha}$, $\tilde{\beta}$.

$p \circ (\widetilde{\alpha * \bar{\beta}}) = \alpha * \bar{\beta}$ & $\widetilde{\alpha * \bar{\beta}}(0) = e_0$

$\tilde{\alpha}(t) = \widetilde{\alpha * \bar{\beta}}(t/2)$

PROOF Let $\widetilde{\alpha * \bar{\beta}}$ be the unique lift of $\alpha * \bar{\beta}$ with $\widetilde{\alpha * \bar{\beta}}(0) = e_0$. We have seen that $\widetilde{\alpha * \bar{\beta}}(1) = e_0$ if $[\alpha * \bar{\beta}] \in p_*(\pi_1(E, e_0))$.
In fact if $\widetilde{\alpha * \bar{\beta}}(1) = e_0$ then $[\widetilde{\alpha * \bar{\beta}}] \in \pi_1(E, e_0)$ & $[\alpha * \bar{\beta}] = [p \circ \widetilde{\alpha * \bar{\beta}}] = p_*([\widetilde{\alpha * \bar{\beta}}])$ so $[\alpha * \bar{\beta}] \in p_*(\pi_1(E, e_0))$.

Now assume $[\alpha * \bar{\beta}] \in p_*(\pi_1(E, e_0))$ and let $\tilde{\alpha}(t) = \widetilde{\alpha * \bar{\beta}}(2t)$ and $\tilde{\beta}(t) = \widetilde{\alpha * \bar{\beta}}(1 - \frac{1}{2}t)$.

Then $\tilde{\alpha}$ & $\tilde{\beta}$ are the unique lifts of α & β with $\tilde{\alpha}(0) = \tilde{\beta}(0) = e_0$.

Therefore $\tilde{\beta}(1) = \widetilde{\alpha * \bar{\beta}}(1/2) = \tilde{\alpha}(1)$.

Now assume $\tilde{\alpha}$ & $\tilde{\beta}$ are the lifts of α & β and $\tilde{\alpha}(1) = \tilde{\beta}(1)$. Then $\widetilde{\alpha * \bar{\beta}}(t) = \tilde{\alpha} * \bar{\tilde{\beta}}(t)$ is the unique lift of $\alpha * \bar{\beta}$ with $\widetilde{\alpha * \bar{\beta}}(0) = e_0$. But $\widetilde{\alpha * \bar{\beta}}(1) = \bar{\tilde{\beta}}(1) = \tilde{\beta}(0) = e_0$.

By the prvs lemma this implies $[\alpha * \bar{\beta}] \in p_*(\pi_1(E, e_0))$

$\boxed{\text{PROOF OF FLL}}$ We define $\tilde{f}(x)$ as before.

Let $\alpha : [0,1] \to X$ be a path with $\alpha(0) = x_0$ & $\alpha(1) = x$. Let $\tilde{\alpha} : [0,1] \to E$ be the unique lift of $f \circ \alpha : [0,1] \to B$ with $\tilde{\alpha}(0) = e_0$ & define $\tilde{f}(x) = \tilde{\alpha}(1)$. To show that this is well defined we let

$$\beta : [0,1] \to X$$

be another path with $\beta(0) = x_0$ & $\beta(1) = x$.

Then $[\alpha * \bar{\beta}] \in \pi_1(X, x_0)$

Then $f_*([\alpha * \bar{\beta}]) = [f \circ (\alpha * \bar{\beta})]$

$$= [(f \circ \alpha) * (\overline{f \circ \beta})] \in P_*(\pi_1(E, e_0)).$$

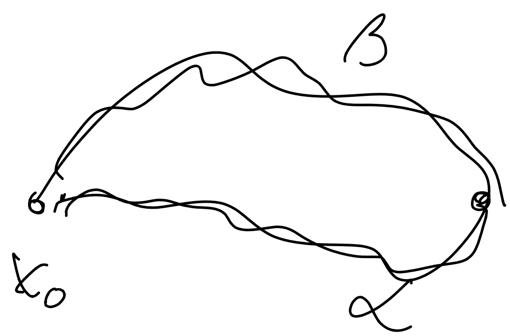
Therefore, by the lemma, $\tilde{\alpha}(1) = \tilde{\beta}(1)$ and $\tilde{f}(x)$ is well defined.

The proof of continuity of $\tilde{f}$ is exactly as before.

Assume $\exists\, [g] \in \pi_1(X, x_0)$ s.t.
$f_*([g]) \notin P_*(\pi_1(E, e_0))$.

$$\alpha(t) = g(t/2)$$

$$\beta(t) = g\left(t \cdot \tfrac{1}{2} + (1-t)\, 1\right)$$



$\Rightarrow \quad \tilde{\alpha}(1) \neq \tilde{\beta}(1)$