COVERING SPACES Let $p: E \to B$ be continuous and surjective. Then a nbd. U is evenly covered if $p^{-1}(U)$ is a disjoint union of open sets V_α such that restricted to each V_α, p is a homeomorphism to U.

p is a covering map if every point in B has an evenly covered nbd.
E is a covering space.

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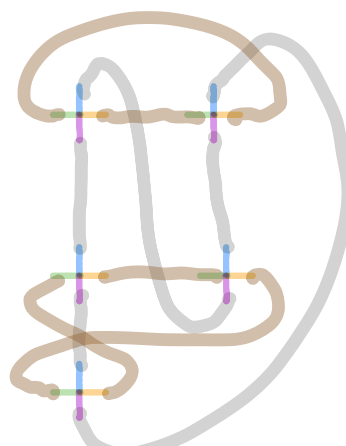
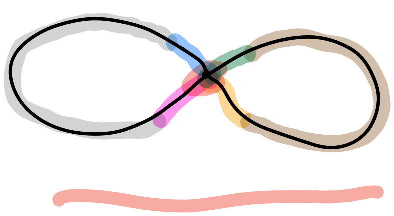
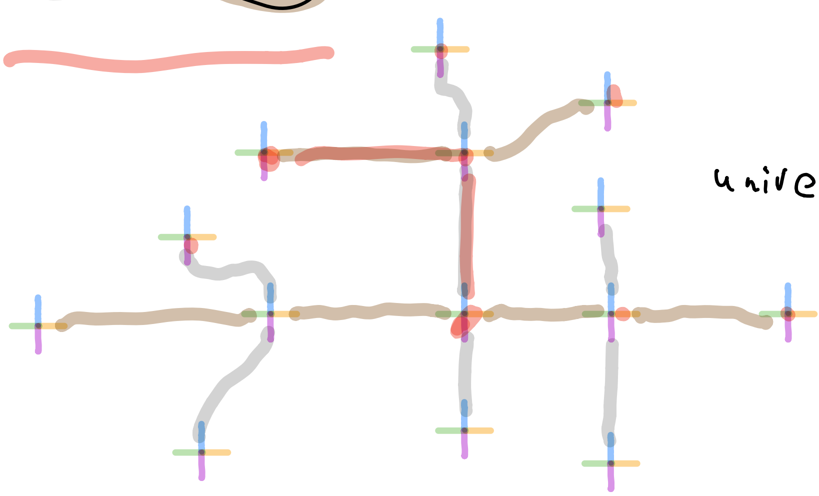
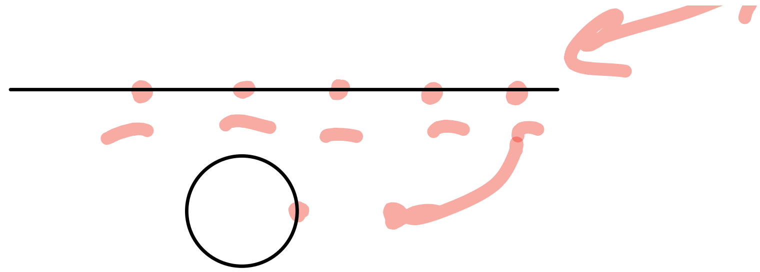
EXAMPLES

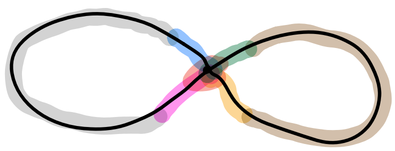






universal cover



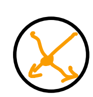


$\boxed{\text{GROUP ACTION}}$ $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

$\phi: S^2 \to S^2$ given by $\phi(x,y,z) = (-x,-y,-z)$.

$(x,y,z) \sim \phi(x,y,z) = (-x,-y,-z)$. antipodal map

$\mathbb{RP}^2 = S^2/\sim$

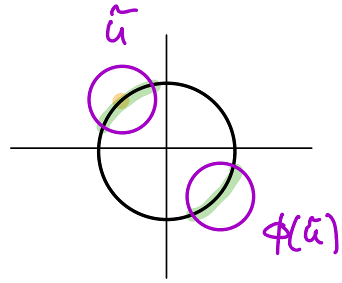


The map $p: S^2 \to \mathbb{RP}^2$ is a covering map.

Note that $\phi^2 = id$.

Need to find evenly covered nbds for every equivalence class $[(x,y,z)] \in \mathbb{RP}^2$.

Let U be a ball of radius $1/2$ centered at $(x,y,z) \in S^2$. Then $\tilde{U} \cap \phi(\tilde{U}) = \emptyset$.



Let $\tilde{V} = \tilde{U} \cap S^2$ & let $V = p(\tilde{V})$ be the image of $\tilde{V}$ in $\mathbb{RP}^2$. Then $p^{-1}(V) = \tilde{V} \cup \phi(\tilde{V})$ so V is evenly covered.

Similar argument work for S^n covering $\mathbb{RP}^n$

$$S^n : \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

$\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $\phi(x_1, \dots, x_{n+1}) = (-x_1, \dots, -x_{n+1})$

$$S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$$

$$\varphi_{n,m}(z,w) = \left(e^{\frac{2\pi}{n} i} z,\ e^{\frac{2\pi}{n} i} w\right)$$

$\varphi_{n,m} : \mathbb{C}^2 \to \mathbb{C}^2$ with $\phi_n(z,w) \in S^3$ iff $(z,w) \in S^3$

$z = x+iy \qquad w = u+iv$

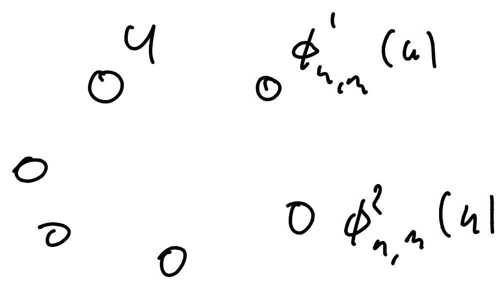
$|z|^2 = x^2+y^2 \quad , |w|^2 = u^2+v^2$

$\Rightarrow \quad |z|^2+|w|^2 = 1$ is the same as $x^2+y^2+u^2+v^2 = 1$

$p: \; S^3 \to S^3/\sim \; \cong L_{n,m}$

p is a covering map.

pf: choose a small ball U around (z,w) s.t. all $\phi^k_{n,m}(U) \cap \phi^j_{n,m}(U) = \emptyset$ if $k \neq j$.



$(z,w) \sim \phi^k_{n,m}(z,w) \quad \forall k$

$\boxed{\text{LIFTING LEMMAS}}$

$p : E \to B$ a covering spaces.

Given $f : X \to B$ when can we find a lift $\tilde{f} : X \to E$?

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd} \qquad f = p \circ \tilde{f}$$

The identity map $id : S^1 \to S^1$ doesn't lift to $\mathbb{R}$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[ur, "\times ?"] \arrow[r, "id"'] & S^1 \end{tikzcd}$$

Why not?

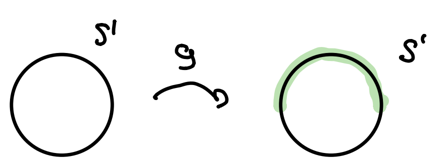
$$\begin{tikzcd} & & \mathbb{R} \arrow[d, "\pi"] \\ {[0,1]} \arrow[urr, bend left, "\tilde{f}_1 = \tilde{id} \circ f_1"] \arrow[r, "f_1"'] & S^1 \arrow[ur, "\tilde{id}"] \arrow[r, "id"'] & S^1 \end{tikzcd}$$

$\tilde{f}_1$ is unique & $\tilde{f}_1(0) \neq \tilde{f}_1(1)$. But $f_1(0) = f_1(1)$ so $\tilde{id} \circ f_1(0) = \tilde{id} \circ f_1(1)$. Contradiction

A constant map $S^1 \to [0] \subset S^1$ will lift.

Constant maps always lift.

$g : S^1 \to S^1$ by $g([t]) = \begin{cases} [t] & 0 \le t \le 1/2 \\ [1-t] & 1/2 < t < 1 \end{cases}$



$$\tilde{g}(t) = \begin{cases} t & 0 \leq t \leq 1/2 \\ 1-t & 1/2 < t < 1 \end{cases}$$ is a lift of g

Lifting criteria cannot just depend on X.