Homework 2, Math 5520
Spring 2018

Let $\{C_i\}$ be a collection of abelian groups indexed by the non-negative integers and $\partial_i \colon C_i \to C_{i-1}$ homomorphisms with $\partial_i \circ \partial_{i+1} = 0$. For convenience we assume that $\partial_0 = 0$. Then $C = \{(C_i, \partial_i)\}$ is a *chain complex*. For each i we define subgroups of C_i by

$$Z_i(C) = \ker \partial_i \text{ and } B_i(C) = \mathrm{im}\partial_{i+1}.$$

The Z_i is the subgroup of *cycles* and B_i is the subgroup of *boundaries*. An element of C_i is a *chain*. The condition that $\partial_i \circ \partial_{i+1} = 0$ implies that $B_i(C) \subset Z_i(C)$. We then define the homology groups by

$$H_i(C) = Z_i(C)/B_i(C).$$

Two cycles $z_0, z_1 \in Z_i(C)$ are *homologous* if they differ by a boundary; that is there exists a $b \in B_i(C)$ such that $b = z_0 - z_1$ or, equivalently, there exists a chain $c \in C_{i+1}$ such that $\partial_{i+1} c = z_0 - z_1$. If $z \in Z_i(C)$ is a cycle then $[z]$ will represent the homology class in $H_i(C)$.

Lets begin with some examples. In what follows assume that $C_i = 0$ for $i > 2$ and $i = 0$ and $C_1 = C_2 = \mathbb{Z}$. Then we must have $\partial_i = 0$ if $i \neq 2$.

1. Calculate $H_2(C)$, $H_1(C)$ and $H_0(C)$ if
   (a) ∂_2 is an isomorphism;
   (b) ∂_2 is the zero map;
   (c) ∂_2 is multiplication by n.

A *chain map* $\phi \colon A \to C$ be chain complexes A and C is a collection of homomorphisms $\phi_i \colon A_i \to C_i$ such that $\phi_{i-1} \circ \partial_i = \partial_i \circ \phi_i$.

2. Show that $\phi_i(Z_i(A)) \subset Z_i(C)$.
3. Show that if $z_0, z_1 \in Z_i(A)$ are homologous then $\phi_i(z_0)$ and $\phi_i(z_1)$ are homologous.
4. Show that there is a well defined homomorphism $(\phi_i)_* \colon H_i(A) \to H_i(C)$ given by $(\phi_i)_*([z]) = [\phi_i(z)]$.

Now let A_i be family of abelian groups and $\phi_i \colon A_i \to A_{i-1}$ homomorphisms. This sequence is *exact* if $\mathrm{im}\phi_i = \ker \phi_{i+1}$. A sequence that is indexed by non-negatives integers is typically called a *long exact sequence*. A sequence of length five where the starting and ending groups are trivial is a *short exact sequence*. If A, B and C are chain complexes and $\phi\colon A \to B$ and $\psi\colon B \to C$ are chain maps then

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

is a short exact sequence of chain complexes if for each i we have that

$$0 \longrightarrow A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \longrightarrow 0$$

is a short exact sequence.

A fundamental result is that a short exact sequence of chain complexes determines a long exact sequence of homology groups. This is called the "snake lemma" and the proof follows from "diagram chasing".

5. Show that $\mathrm{im}(\phi_i)_* \subset \ker(\psi_i)_*$.

6. Show that if $\beta \in B_i$ is a cycle and $\psi_i(\beta) = 0$ then there exists a cycle $\alpha \in A_i$ with $\phi_i(\alpha) = \beta$. Conclude that $\mathrm{im}(\phi_i)_* = \ker(\psi_i)_*$.

7. Given a cycle $\gamma \in C_i$ show that there exists a chain $\beta \in B_i$ with $\psi_i(\beta) = \gamma$ and a $\alpha \in A_{i-1}$ such that $\phi_{i-1}(\alpha) = \partial_i\beta$.

For the below problems assume that $\alpha \in A_{i-1}$, $\beta \in B_i$ and $\gamma \in C_i$ with $\phi_{i-1}(\alpha) = \partial_i\beta$ and $\psi_i(\beta) = \gamma$.

8. Show that if $\gamma = 0$ then α is a boundary. Conclude that if $\beta_0, \beta_1 \in B_i$ with $\psi_i(\beta_j) = \gamma$ and $\alpha_0, \alpha_i \in A_{i-1}$ with $\phi_{i-1}(\alpha_j) = \partial_i\beta_j$ then α_0 and α_1 are homologous.

9. If γ is a boundary show that β can be chosen to be a boundary. Use this and (8) to show that if $\gamma_0, \gamma_1 \in C_i$ are homologous, $\beta_0, \beta_1 \in B_i$ with $\psi_i(\beta_j) = \gamma_j$ and $\alpha_0, \alpha_1 \in A_{i-1}$ with $\phi_i(\alpha_j) = \partial_i\beta_j$ then α_0 and α_1 are homologous.

10. Conclude that there is a well defined homomorphism $\delta_i\colon H_i(C) \to H_{i-1}(A)$ given by $\delta_i([\gamma]) = [\alpha]$.

11. If β is a cycle show that $\alpha = 0$ and conclude that $\mathrm{im}(\psi_i)_* \subset \ker\delta_i$.

12. If α is a boundary show that β can be chosen to be a cycle. Conclude that $\ker\delta_i \subset \mathrm{im}(\psi_i)_*$ and therefore $\ker\delta_i = \mathrm{im}(\psi_i)_*$.

13. By the definition of α we have that $\psi_{i-1}(\alpha) = \partial_i\beta$ is a boundary. Conclude that $\mathrm{im}\delta_i \subset \ker(\psi_{i-1})_*$.

14. Given a cycle $\alpha' \in A_{i-1}$ such that $\phi_{i-1}(\alpha')$ is a boundary show that there exists a $\beta' \in B_i$ and a cycle $\gamma' \in C_i$ with $\psi_i(\beta') = \gamma'$ and $\phi_{i-1}(\alpha') = \partial_i \beta'$. Conclude that $\ker(\psi_{i-1})_* \subset \mathrm{im}\delta_i$ and therefore $\ker(\psi_{i-1})_* = \mathrm{im}\delta_i$.

Congratulations! You have proved the snake lemma!

There are some important examples. Let C be a a chain complex. If $B_i \subset C_i$ are subgroups with $\partial_i(B_i) \subset B_{i-1}$ then $B = \{(B_i, \partial_i)\}$ is a sub-chain complex. The quotient groups C_i/B_i also from a chain complex:

15. Let $c_0, c_1 \in C_i$ be chains such that $c_1 - c_0 \in B_i$. Show that $\partial_i c_0 - \partial_i c_1 \in B_{i-1}$. Conclude that ∂_i descends to a map $C_i/B_i \to C_{i-1}/B_{i-1}$.

16. Show that
$$0 \longrightarrow B \longrightarrow C \longrightarrow C/B \longrightarrow 0$$
is a short exact sequence of chain complexes.

Another natural example comes from a chain complex C and two subcomplexes $A, B \subset C$ such that for each i, A_i and B_i generate C_i. That is ever element c can be written as a sum $c = a + b$ where $a \in A_i$ and $b \in B_i$.

17. Let $D_i = A_i \cap B_i$ and show that $D = \{(D_i, \partial_i)\}$ is a subcomplex of C.

18. Show that $A \bigoplus B = \{(A_i \bigoplus B_i, \partial_i \oplus \partial_i)\}$ is a chain complex.

19. Let ι_A and ι_B be the inclusion maps of D in A and B, respectively. Show that these are chain maps and the map $D \to A \bigoplus B$ given by $d \mapsto (\iota_A(d), -\iota_B(d))$ is a chain map.

20. Let j_A and j_B be the inclusion maps of A and B into C. Show that the map $(a, b) \mapsto j_A(a) + j_B(b)$ is a chain map from $A \bigoplus B \to C$.

21. Show that
$$0 \longrightarrow D \overset{(\iota_A, -\iota_B)}{\longrightarrow} A \bigoplus B \overset{j_A + j_B}{\longrightarrow} C \longrightarrow 0$$
is a short exact sequence of chain complexes.

Simplicial complexes

Let $\mathcal{S}$ be a set. Then $\mathbb{Z}(\mathcal{S})$ is the group of *formal sums* of $\mathcal{S}$ with $\mathbb{Z}$-coefficients. That is an element of $\mathbf{n} \in \mathbb{Z}(\mathcal{S})$ is an assignment to each $s \in \mathcal{S}$ and integer n_s such that at but finitely many of the coefficients in $\mathbf{n}$ are zero. The group operation is then adding coefficients.

If $\mathcal{S}$ is a finite set (as it will be for our examples) then the last condition automatically holds. However, there are many natural situations (often arising in topology) where $\mathcal{S}$ can be an infinite set. One can also replace the group $\mathbb{Z}$ with an arbitrary group. Common examples are $\mathbb{R}$ or more generally an arbitrary field but we will stick to $\mathbb{Z}$.

1. Show that $\mathbb{Z}(\mathcal{S})$ is a group.
2. Let $\mathcal{R}$ be another set. Show that any map of $\mathbb{R}$ to $\mathbb{Z}(\mathcal{S})$ extends to a unique homomorphism from $\mathbb{Z}(\mathcal{R})$ to $\mathbb{Z}(\mathcal{S})$.

Now let $\mathcal{S}$ be a finite ordered set with $n+1$ elements. Let $\mathcal{S}^{(k)}$ be the set of subsets of $\mathcal{S}$ with $k+1$ elements. Note that $\mathcal{S}^{(k)}$ will have $\begin{pmatrix} n+1 \\ k+1 \end{pmatrix}$ elements.

Let $\{v_0, \ldots, v_k\}$ be an element in $\mathcal{S}^{(k)}$, where the indices indicate the order, and define $\partial_k \colon \mathcal{S}^{(k)} \to \mathbb{Z}\left(\mathcal{S}^{(k-1)}\right)$ by

$$\partial_k\{v_0, \ldots, v_k\} = \sum_{i=0}^{k} (-1)^i \{v_0, \ldots, \hat{v}_i, \ldots, v_k\}.$$

Here, the $\hat{v}_i$ indicates that v_i has been removed from the set.

By (2) this extends to a homomorphism $\partial_k \colon \mathbb{Z}\left(\mathcal{S}^{(k)}\right) \to \mathbb{Z}\left(\mathcal{S}^{(k-1)}\right)$.

3. Show that $\partial_{k-1} \circ \partial_k = 0$ so that $\{\mathbb{Z}(\mathcal{S}^{(k)}), \partial_k\}$ is a chain complex.

Now let X be a collection of subsets of $\mathcal{S}$ with the property that if $A \in X$ and B is a subset of A then $B \in X$. Then X is an *abstract simplicial complex*. We let $X^{(k)} = X \cap \mathcal{S}^{(k)}$ be those subsets in X that have $k+1$ elements.

4. If X is an abstract simplicial complex show that $\partial_k\left(\mathbb{Z}\left(X^{(k)}\right)\right) \subset \mathbb{Z}\left(X^{(k-1)}\right)$ and therefore $\{\mathbb{Z}\left(X^{(k)}\right), \partial_k\}$ is sub-chain complex of $\{\mathbb{Z}\left(\mathcal{S}^{(k)}\right), \partial_k\}$.

A *topological simplicial complex* is a topological space X that is a union of simplices such that the intersection of any two simplices is a single simplex.

5. Let X be a topological simplicial complex. Let $X^{(0)}$ be the set of vertices of X and give this set an order. Let $X^{(k)}$ be the subsets of $X^{(0)}$ with $k+1$ elements that span a k-simplex in X. Show that $\cup X^{(k)}$ is an abstract simplicial complex.
6. Show that any abstract simplicial complex can be realized as a topological simplicial complex.