Notes and problems on the topology of $\mathbb{R}^n$

Let X be a set and $d : X \times X \to [0, \infty)$ a function with:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) + d(y, z) \geq d(x, z)$.

Then d is a *metric* on X and the pair (X, d) is a *metric space.* Property (3) is the *triangle inequality.*

Define a $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by setting

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$.

Problem 1 Show that d is a metric on $\mathbb{R}^n$.

The open ball of radius r centered at $\mathbf{x}$ is the set

$$B_r(\mathbf{x}) = \{\mathbf{y} | d(\mathbf{x}, \mathbf{y}) < r\}.$$

The triangle inequality implies that if $r_0 < r_1$ then $B_{r_0}(\mathbf{x}) \subset B_{r_1}(\mathbf{x})$.

A subset $U \subset \mathbb{R}^n$ is *open* if for every $\mathbf{x} \in U$ there is an $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset U$.

Theorem 1 *The open subsets of $\mathbb{R}^n$ satisfy the following properties:*

1. *$\mathbb{R}^n$ and $\emptyset$ are open.*
2. *If $\{U_\alpha\}$ is a collection of open sets then $\bigcup U_\alpha$ is open.*
3. *If $U_1, \ldots, U_n$ are open then $\bigcap U_i$ is open.*

Proof of 1. Obvious.

2. If $\mathbf{x} \in \bigcup U_\alpha$ then $\mathbf{x} \in U_\alpha$ for some α. Since U_α is open there exists an ϵ such that $B_\epsilon(\mathbf{x}) \subset U_\alpha$. But U_α is contained in $\bigcup U_\alpha$ so we also have $B_\epsilon(\mathbf{x}) \subset \bigcup U_\alpha$ and $\bigcup U_\alpha$ is open.

3. If $\mathbf{x} \in \bigcap U_i$ then $\mathbf{x} \in U_i$ for all $i = 1, \ldots, n$ so there exists ϵ_i with $B_{\epsilon_i}(\mathbf{x}) \subset U_i$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Since $B_\epsilon(\mathbf{x}) \subset B_{\epsilon_i}(\mathbf{x})$ for all $i = 1, \ldots, n$ we have $B_\epsilon(\mathbf{x}) \subset U_i$ for all i. Therefore $B_\epsilon(\mathbf{x}) \subset \bigcap U_i$ and $\bigcap U_i$ is open. $\boxed{1}$

A subset U of $\mathbb{R}^n$ is *closed* if $U^c = \mathbb{R}^n \backslash U$ is open.

Problem 2 Prove that the closed subsets of $\mathbb{R}^n$ satisfy the following properties:

1. $\mathbb{R}^n$ and $\emptyset$ are closed.
2. If $\{U_\alpha\}$ is a collection of closed sets then $\bigcap U_\alpha$ is closed.
3. If $U_1, \dots, U_n$ are closed then $\bigcup U_i$ is closed.

Here is another characterization of a closed set.

Theorem 2 *A set U is closed if and only if for every sequence $\{\mathbf{x}_i\}$ in U with $\mathbf{x}_i$ converging to some $\mathbf{x} \in \mathbb{R}^n$ then $\mathbf{x} \in U$.*

The *interior* of a set U, denoted int$U$, is the union of all open set contained in U.

Problem 3 Show that

$$\text{int}U = \{x \in U | \text{ there exists } \epsilon > 0 \text{ with } B_\epsilon(\mathbf{x}) \subset U\}.$$

The closure of U, denoted $\bar{U}$, is the intersection of all closed sets that contain U. Let A be a subset of B. Then A is *dense* in B if $\bar{A} \supset B$.

Problem 4 Show that $\mathbb{Q}$ is dense in $\mathbb{R}$. More generally show that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Let $B_\mathbb{Q}$ be the collection of balls $B_r(\mathbf{x})$ with $\mathbf{x} \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$.

Problem 5 Show that $B_\mathbb{Q}$ is countable.

Theorem 3 *If U is an open set define*

$$U_\mathbb{Q} = \bigcup_{B \in B_\mathbb{Q} \text{ and } B \subset U} B.$$

Then $U = U_\mathbb{Q}$.

Proof. Clearly $U_\mathbb{Q} \subset U$ so we only need to show that $U \subset U_\mathbb{Q}$. If $\mathbf{x} \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}^n$ there exists $\mathbf{y} \in \mathbb{Q}^n \cap B_{\epsilon/3}(\mathbf{x})$. Again using the density of $\mathbb{Q}$ in $\mathbb{R}$ we can find an $r \in (\epsilon/3, \epsilon/2) \cap \mathbb{Q}$. Then $B_r(\mathbf{y}) \in B_\mathbb{Q}$. Since $d(\mathbf{x}, \mathbf{y}) \leq \epsilon/3$ we also have $\mathbf{x} \in B_r(\mathbf{y})$. Furthermore if $\mathbf{z} \in B_r(\mathbf{y})$ then by the triangle inequality

$$d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \leq \epsilon/3 + r \leq \epsilon/3 + \epsilon/2 < \epsilon$$

and therefore $B_r(\mathbf{y}) \subset B_\epsilon(\mathbf{x}) \subset U$. Hence $\mathbf{x} \in U_\mathbb{Q}$ and $U \subset U_\mathbb{Q}$ as desired. $\boxed{3}$