Notes and problems on the Riemann integral

We recall the definition of the Riemann integral.

A *partition* P of an interval $[a, b]$ is a finite sequence $x_0 = a < x_1 < \cdots < x_n = b$.

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. We define the *lower sum of* f *with respect to the partition* P as follows. Let

$$m_i = \inf_{[x_{i-1}, x_i]} f.$$

Then the lower sum is defined by

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

We similarly define the *upper sum of* f *with respect to* P by

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

where

$$M_i = \sup_{[x_{i-1}, x_i]} f.$$

Note that $m_i \leq M_i$ for all i and therefore $L(f, P) \leq U(f, P)$.

Let $\mathcal{P}$ be the set of all partitions of $[a, b]$. Then the *lower integral* of f is defined by

$$L_a^b(f) = \sup_{P \in \mathcal{P}} L(f, P)$$

and the *upper integral* of f is defined by

$$U_a^b(f) = \inf_{P \in \mathcal{P}} U(f, P).$$

The function f is *Riemann integrable* if $L_a^b(f) = U_a^b(f)$ and the Riemann integral of f is

$$\int_a^b f(x)dx = L_a^b(f) = U_a^b(f).$$

Problems.

1. A partition P' is a *refinement* of P if $P \subseteq P'$. For any refinement P' of P show that $L(f, P') \geq L(f, P)$ and $U(f, P') \leq U(f, P)$.

2. For any two partitions P and Q show that $L(f,P) \leq U(f,Q)$ and therefore $L_a^b(f) \leq U_a^b(f)$. (Hint: Compare the two sums to the upper and lower sums of a common refinement of P and Q and then use the previous problem.)

3. Show that f is Riemann integrable if and only if for all $\epsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) < \epsilon$.

4. Show that f is Riemann integrable if and only if there exists a sequence of partitions P_i such that $U(f,P_i) - L(f,P_i) \to 0$ as $i \to \infty$. If f is integrable show that

$$\int_a^b f(x)dx = \lim_{i \to \infty} L(f,P_i).$$

5. Let $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$ with $x_i - x_{i-1} < w$ for all $i = 1, \ldots n$.

   (a) If $f(x) = c_1 x + c_2$ show that $U(f,P) - L(f,P) \leq w|c_1|(b-a)$.

   (b) If f is a differentiable function on $[a,b]$ with $|f'(x)| \leq c$ for all $x \in [a,b]$ show that $U(f,P) - L(f,P) \leq wc(b-a)$.