Math 6220
Homework 2
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Problem 1.2.4.2

The vertices of the cube on the unit sphere are the points $(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$. Using the formula $(x, y, t) \to z = \frac{x+iy}{1-t}$, we see that the 8 vertices are mapped to $\frac{\pm\frac{1}{\sqrt{3}} \pm i\frac{1}{\sqrt{3}}}{1 \pm \frac{1}{\sqrt{3}}}$.

Problem 2.1.2.1 We want to show that the composition of two differentiable functions has a derivative. Assume first that f is differentiable at a point z_0 and g is differentiable at the point $w_0 = f(z_0)$, and then define the following function:

$$h(w) = \begin{cases} \frac{g(w)-g(w_0)}{w-w_0}, & \textbf{if } w \neq w_0 \\ g'(w_0), & \textbf{if } w = w_0 \end{cases}$$

$h(w)$ is continuous when $w \neq w$, since g is differentiable, and it is also continuous at $w = w_0$ since by definition $g'(w_0) = \lim_{w \to w_0} \frac{g(w)-g(w_0)}{w-w_0}$. Next, one may write the expression

$$g(w) - g(w_0) = h(w)(w - w_0).$$

Making the subsitution $w_0 \mapsto f(z_0)$ and $w \mapsto f(z)$ and dividing both sides by $z - z_0 (z \neq z_0)$ yields

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = h(f(z))\frac{f(z) - f(z_0)}{z - z_0}.$$

We may take the limit of both sides of the above, as $z \to z_0$, since h is a continuous function and because $g(z)$ is analytic. This yields the expression

$$\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = \lim_{z \to z_0} h(g(z)) \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0},$$

where the limit distributes on the right hand side because both limits exist and are well defined. Using the definition of the derivative, the above expression is equivalent to $g(f(z))' = g'(f(z))f'(z)$, and of course since $g'(f(z))$ and $f'(z)$ are well defined, f composed with g has a well defined derivative and thus is analytic.

Problem 2.1.2.3
Put $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$**. By calculation,**

$$\begin{array}{lcl} u_x = 3ax^2 + 2bxy + cy^2 & , & u_{xx} = 6ax + 2by \\ u_y = 3dy^2 + 2cxy + bx^2 & , & u_{yy} = 6dy + 2cx \end{array} .$$

By the requirment that u **is harmonic,** $\triangle u$ **must be** 0 **or** $u_{xx} + u_{yy} = 0$ **for all** $x, y \in \mathbb{R}$**. It follows that** $c = -3a$ **and** $b = -3d$ **and** u **is rewritten as**

$$u(x,y) = ax^3 - 3dx^2y - 3axy^2 + dy^3 \qquad a, d \in \mathbb{R}$$

Now we determine v**, the conjugate harmonic function of** u
(i) By integration
Since $v_y = u_x = 3ax^2 - 6dxy - 3ay^2$**,** v **has the form**

$$v(x,y) = 3ax^2y - 3dxy^2 - ay^3 + C(x)$$

and since $v_x = -u_y$**,**

$$6axy - 3dy^2 + C'(x) = -3dy^2 + 6axy + 3dx^2$$

therefore, $C(x) = dx^3 + C$**. It follows**

$$v(x,y) = dx^3 + 3ax^2y - 3dxy^2 - ay^3 + C$$

(ii) We find v **by the fact that** $f(z) = u + iv = 2u(\frac{z}{2}, \frac{z}{2i})$**.**

$$\begin{aligned} 2u(\frac{z}{2}, \frac{z}{2i}) &= 2(a\frac{z^3}{8} - 3d\frac{z^2}{4}\frac{z}{2i} - 3a\frac{z}{2}\frac{z^2}{4i^2} + d\frac{z^3}{8i^3}) \\ &= \frac{z^3}{4}(a + 3di + 3a + di) \\ &= (x^3 + 3ix^2y + 3xi^2y^2 + i^3y^3)(a + di) \\ &= (x^3 - 3xy^2 + i(3x^2y - y^3))(a + di) \\ &= 3x^3 - 3axy^2 - 3dx^2y + dy^3 + i(dx^3 - 3dxy^2 + 3ax^2y - ay^3) \end{aligned}$$

Hence, $v(x,y) = dx^3 + 3ax^2y - 3dxy^2 - ay^3 + C$**.**

Problem 2.1.2.4
Assume that f **is analytic and** $|f(z)| = c$ **for all** z **where** $c \neq 0 \in \mathbb{R}$**. Note that you can assume that** $c \neq 0$**, since** $|f(z)| = 0$ **implies that** $z = 0$**. Notice that**

$$|f(z)|^2 = c^2 \;\Leftrightarrow\; f(z)\overline{f(z)} = c^2 \;\Leftrightarrow\; \overline{f(z)} = \frac{c^2}{f(z)}.$$

This shows that $\overline{f}$ is analytic as long as f is analytic and nonzero. From this you can conclude that both f and $\bar{f}$ satisfy the Cauchy-Riemann equations. Let $f(x,y) = u(x,y) + iv(x,y)$. You have that

$$u_x = \quad v_y \qquad\qquad u_x = -v_y$$
$$u_y = -v_x \qquad\qquad u_y = \quad v_x.$$

The above implies that $u_x = u_y = 0$ and $v_x = v_y = 0$. Integrating u_x with respect to x yields:

$$u(x,y) = \int 0 \, dx = \varphi(y)$$

where φ is some real valued function of y. Now, differentiating with respect to y gives you $u_y = \varphi'(y)$. Since this must be zero, you have that $\varphi(y) = a$ where $a \in \mathbb{R}$ and hence, $u(x,y) = a$. Using a similiar arguement, you can show that $v(x,y) = b$ where $b \in \mathbb{R}$. Therefore, $f(z) = a + bi$.

Problem 2.1.2.7 Show that a harmonic function satisfies the formal differential equation

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} = 0.$$

Let u be harmonic. Thus, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Using the definitions for $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ on page 27 of Ahlfors, we have

$$\frac{\partial u}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right),$$

and so

$$\begin{aligned}
\frac{\partial^2 u}{\partial z \partial \overline{z}} &= \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \overline{z}}\right) \\
&= \frac{\partial}{\partial z}\left(\frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)\right) \\
&= \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \\
&= \frac{1}{2}\cdot\frac{1}{2}\left(\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right] - i\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right]\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 u}{\partial x \partial y} - i\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2}\right) \\
&= 0
\end{aligned}$$

since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Problem 2.1.4.2

Since there are n distinct roots, then each root, α_i, is called a simple zero and is characterized by the condition $Q(\alpha_i) = 0$ and $Q'(\alpha_i) \neq 0$. So we have

$$\frac{P(z)}{Q(z)} = \frac{P(z)}{(z-\alpha_1)...(z-\alpha_n)} = \frac{A_1}{(z-\alpha_1)} + \dots \frac{A_n}{(z-\alpha_n)}$$

for $A_1, \dots A_n$ unknown. Now notice,

$$\begin{aligned}
Q'(z) \quad &= [(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_{i-1})(z-\alpha_i)(z-\alpha_{i+1})\dots(z-\alpha_n)]' \\
&= [(z-\alpha_i)(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_{i-1})(z-\alpha_{i+1})\dots(z-\alpha_n)]' \\
&= (z-\alpha_i)'[(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_{i-1})(z-\alpha_{i+1})\dots(z-\alpha_n)] \\
&\quad +[(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_{i-1})(z-\alpha_{i+1})\dots(z-\alpha_n)]'(z-\alpha_i) \\
\Rightarrow Q'(\alpha_i) \quad &= (\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\dots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\dots(\alpha_i-\alpha_n) \qquad (\Diamond)
\end{aligned}$$

So we have

$$\begin{aligned}
\frac{P(z)}{Q(z)} \quad &= \frac{A_1}{(z-\alpha_1)} + \frac{A_2}{(z-\alpha_2)} \cdots \frac{A_n}{(z-\alpha_n)} \\
\Rightarrow P(z) \quad &= \frac{A_1 Q(z)}{(z-\alpha_1)} + \frac{A_2 Q(z)}{(z-\alpha_2)} \cdots \frac{A_n Q(z)}{(z-\alpha_n)} \\
&= A_1(z-\alpha_2)(z-\alpha_3)\dots(z-\alpha_n) \\
&\quad +A_2(z-\alpha_1)(z-\alpha_3)\dots(z-\alpha_n) \\
&\quad \vdots \\
&\quad +A_n(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_{n-1})
\end{aligned}$$

To solve for A_i we evaluate $P(\alpha_i)$. Notice when we evaluate P at α_i we have

$$\begin{aligned}
P(\alpha_i) \quad &= A_i(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_{i-1})(z-\alpha_{i+1})\dots(z-\alpha_n) \\
&= A_i Q'(\alpha_i) \qquad \text{(by } \Diamond\text{)} \\
\Rightarrow A_i \quad &= \frac{P(\alpha_i)}{Q'(\alpha_i)}
\end{aligned}$$

This is true for every i s.t. $1 \leq i \leq n$ and so our claim is proven.

$$\begin{aligned} P(z) &= \tfrac{P(\alpha_1)}{Q'(\alpha_1)(z-\alpha_1)} + \tfrac{P(\alpha_2)}{Q'(\alpha_2)(z-\alpha_2)} + \cdots \tfrac{P(\alpha_n)}{Q'(\alpha_n)(z-\alpha_n)} \\ &= \textstyle\sum_{i=1}^{n} \tfrac{P(\alpha_i)}{Q'(\alpha_i)(z-\alpha_i)} \end{aligned}$$

Problem 2.1.4.3
Proof: Using the conclusion of 2.1.4.2, let

$$P(Z) = \sum_{k=1}^{n} \frac{C_k}{Q'(\alpha_k)(Z-\alpha_k)} Q(Z)$$

then $P(\alpha_k) = C_k$, and $deg(P(Z)) < n$. This proves the existence of such polynomial.

If there is another polynomial $G(Z)$, satisfying $G(\alpha_k) = C_k$, and $deg(G(Z)) < n$, then $P(Z) - G(Z)$ is a polynomial, whose degree is less than n and has n roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. So $P(Z) - G(Z) = 0$. Hence $P(Z) = G(Z)$, which implies the uniqueness.