

Rational numbers - Definitions and problems

Let m and n be integers with $n \neq 0$. We define a the symbol $\frac{m}{n}$ to be the set of all ordered pairs of integers (a, b) with $b \neq 0$ such that $mb = na$. Here is the same definition of $\frac{m}{n}$ using mathematical symbols:

$$\frac{m}{n} = \{(a, b) | a, b \in \mathbb{Z}, b \neq 0 \text{ and } mb = na\}.$$

We call $\frac{m}{n}$ a *rational number*. The set of all rational numbers is denoted by \mathbb{Q} .

(1) Show that for any integer $k \neq 0$ the ordered pair (kn, km) is in $\frac{n}{m}$.

(2) If $\gcd(n, m) = 1$ show that every ordered pair in $\frac{n}{m}$ is of the form (kn, km) where k is a non-zero integer.

(3) Show that if the ordered pair (a, b) is in $\frac{n}{m}$ then $\frac{a}{b}$ is equal to $\frac{n}{m}$ as sets.

We define the addition of two rational numbers as follows. Define

$$\frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}.$$

We define multiplication by

$$\frac{m}{n} \times \frac{p}{q} = \frac{mp}{nq}.$$

These definitions are very easy to deal with since they are exactly the ones that we are used to. However to be rigorous we need to make sure that they are well defined. In particular by (3) we know that $\frac{1}{2}$ and $\frac{2}{4}$ are the same sets. We need to check that the above rules for addition and multiplication give the same answer wether we use $\frac{1}{2}$ or $\frac{2}{4}$.

Here is an example. By the definition of addition

$$\frac{1}{2} + \frac{2}{3} = \frac{1 \times 3 + 2 \times 2}{2 \times 3} = \frac{7}{6}$$

and

$$\frac{2}{4} + \frac{2}{3} = \frac{2 \times 3 + 2 \times 4}{4 \times 3} = \frac{14}{12}.$$

Since $7 \times 12 = 6 \times 14$ the order pair $(14, 12)$ is in $\frac{7}{6}$. By (3) we have $\frac{7}{6} = \frac{14}{12}$ so in this example we get the same answer using $\frac{1}{2}$ and $\frac{2}{4}$.

(4) Show that if $\frac{m}{n} = \frac{a}{b}$ then

$$\frac{m}{n} + \frac{p}{q} = \frac{a}{b} + \frac{p}{q}$$

for all rational numbers $\frac{p}{q}$.

(5) Show that if $\frac{m}{n} = \frac{a}{b}$ then

$$\frac{m}{n} \times \frac{p}{q} = \frac{a}{b} \times \frac{p}{q}$$

for all rational numbers $\frac{p}{q}$.

Define a function f from \mathbb{Z} to \mathbb{Q} by $f(n) = \frac{n}{1}$. We will use this function in the next two problems.

(6) Show that $f(n + m) = f(n) + f(m)$.

(7) Show that $f(n \times m) = f(n) \times f(m)$.

Solutions

(1) We need to show that the ordered pair (kn, km) is in the set of ordered pairs $\frac{n}{m}$. First we observe that kn and km are integers since the product of two integers is an integer. Second $km \neq 0$ since $k \neq 0$ and $m \neq 0$. Finally $nk m = mkn$ since multiplication is commutative. By definition an ordered pair that satisfies these three properties is in $\frac{n}{m}$.

(2) Let (a, b) be an ordered pair in $\frac{n}{m}$. Then a and b are integers, $b \neq 0$ and $na = mb$. By the last property we see that m is a factor of the integer na . Since $\gcd(n, m) = 1$, the only common factor of n and m is 1. Therefore m must be a factor of a . That is there is a non-zero integer k such that $a = km$. If we replace a in the equation $na = mb$ with km we get the equation $nkm = mb$. Since $m \neq 0$ this implies that $kn = b$.

(3) Let (c, d) be an ordered pair in $\frac{n}{m}$. We will show that (c, d) is also in $\frac{a}{b}$. Since (c, d) is in $\frac{n}{m}$, we know that c and d are integers, $d \neq 0$ and $mc = nd$. To show that (c, d) is in $\frac{a}{b}$ we are only left to show that $bc = ad$. Since (a, b) is in $\frac{n}{m}$ we also have $ma = nb$. Since $mc = nd$ and $ma = nb$ we have $(mc) \times (nb) = (nd) \times (ma)$ which implies that $bc = ad$ as desired. Therefore (c, d) is in $\frac{a}{b}$. We have shown that if (a, b) is in $\frac{n}{m}$ then $\frac{n}{m}$ is a subset of $\frac{a}{b}$.

To finish the proof we note that the ordered pair (n, m) is contained in $\frac{n}{m}$ and is therefore also contained in $\frac{a}{b}$. If (n, m) is an element of $\frac{a}{b}$ we have just shown that $\frac{n}{m}$ is a subset of $\frac{a}{b}$. Since $\frac{a}{b}$ is a subset of $\frac{n}{m}$ and $\frac{n}{m}$ is a subset of $\frac{a}{b}$ we must have that $\frac{n}{m} = \frac{a}{b}$.

(4) By the definition of addition

$$\frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}$$

and

$$\frac{a}{b} + \frac{p}{q} = \frac{aq + bp}{bq}.$$

By (3) if the ordered pair $(mq + np, nq)$ is in the set $\frac{aq+bp}{bq}$ then $\frac{aq+bp}{bq} = \frac{mq+np}{nq}$. We now check that $(mq + np, nq)$ satisfies the three defining properties of $\frac{aq+bp}{bq}$. First we note that $mq + np$ and nq are integers since they are products and sums of integers and that $nq \neq 0$ since neither n nor q are 0. This is the first two properties. Using this fact we have

$$\begin{aligned} (bq) \times (mq + np) &= bmq^2 + bnpq \\ &= naq^2 + bnpq \\ &= (nq) \times (aq + bp) \end{aligned}$$

where we are using the fact that $mb = na$ in the second inequality. This is the third property so $(mq + np, nq)$ is in $\frac{aq+bp}{bq}$. Combining the equalities we have

$$\frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq} = \frac{aq + bp}{bq} = \frac{a}{b} + \frac{p}{q}$$

as desired.

(5) By the definition of multiplication

$$\frac{m}{n} \times \frac{p}{q} = \frac{mp}{nq}$$

and

$$\frac{a}{b} \times \frac{p}{q} = \frac{ap}{bq}.$$

As in (4) we need to show that the ordered pair (mp, nq) is the set $\frac{ap}{bq}$. Checking the first two properties we see that mp and nq are integers since they are the product of integers and that $nq \neq 0$ since neither n nor q are zero. Finally we see that $(bq) \times (mp) = bmqp = anqp = (ap) \times (nq)$ where the second inequality uses the fact that $mb = na$. We have shown that (mp, nq) is in $\frac{ap}{bq}$ so by (3) we have $\frac{mp}{nq} = \frac{ap}{bq}$. Exactly as in (4) combining the equalities gives

$$\frac{m}{n} \times \frac{p}{q} = \frac{a}{b} \times \frac{p}{q}$$

as desired.

(6) By the definition of f , $f(n) + f(m) = \frac{n}{1} + \frac{m}{1}$. By the definition of addition, $\frac{n}{1} + \frac{m}{1} = \frac{n \times 1 + m \times 1}{1 \times 1} = \frac{n+m}{1}$. Again using the definition of f we have $f(n+m) = \frac{n+m}{1}$. Combining the equalities gives $f(n+m) = f(n) + f(m)$.

(7) By the definition of f , $f(n) \times f(m) = \frac{n}{1} \times \frac{m}{1}$. By the definition of multiplication $\frac{n}{1} \times \frac{m}{1} = \frac{n \times m}{1 \times 1} = \frac{n \times m}{1}$. We again use the definition of f to see that $f(n \times m) = \frac{n \times m}{1}$. As in (6) combining the equalities gives $f(n \times m) = f(n) \times f(m)$.