



Rigidity of Geometrically Finite Hyperbolic Cone-Manifolds

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Abstract. In a recent paper Hodgson and Kerckhoff prove a local rigidity theorem for finite volume, three-dimensional hyperbolic cone-manifolds. In this paper we extend this result to geometrically finite cone-manifolds. Our methods also give a new proof of a local version of the classical rigidity theorem for geometrically finite hyperbolic 3-manifolds.

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1. Introduction

A hyperbolic cone-manifold is a singular hyperbolic structure where the singularity is a simple closed curve with cross-section a hyperbolic cone. We say a hyperbolic structure on a manifold, M_0 , is *locally rigid* if for any smooth family of hyperbolic metrics M_t , M_0 is isometric to M_t for small t . In a recent paper [HK] Hodgson and Kerckhoff prove a local rigidity result for finite volume three-dimensional hyperbolic cone-manifolds. In this paper we extend this result to geometrically finite cone-manifolds without rank one cusps. The methods employed were first developed by Calabi [Cal] and Weil [Wel] in their proof that closed hyperbolic manifolds of dimension ≥ 3 are locally rigid. Garland [Gar] extended their result to finite volume hyperbolic manifolds.

The result here is another example of the rich deformation theory of hyperbolic manifolds that is special to dimension three. The contrast between three-dimensional hyperbolic manifolds and those of dimension ≥ 4 can be seen in Garland and Raghunathan's [GR] proof that finite volume hyperbolic manifolds of dimension ≥ 4 cannot be deformed even through incomplete hyperbolic structures while in dimension 3 Thurston [Th] showed that if M has nonempty boundary there is at least a one-dimensional space of deformations through incomplete structures. The basic philosophy is that a hyperbolic structure is determined by its boundary geometry. We will exploit Thurston's result in an essential way here.

A geometrically finite hyperbolic structure on the interior of a 3-manifold M extends to a conformal structure on the boundary of M . Our work here provides

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a new proof of the following well known result which is the work of many people including Ahlfors, Bers, Kra, Marden, Maskit, Mostow and Prasad. An expository account can be found in [Bers].

THEOREM 1.1. *M is locally rigid rel the conformal boundary.*

If ∂M is incompressible then the classical proof shows that this is a global result. In the general case there is also a global result although it takes more work to state.

Although the result is stronger than the local theorem we prove the methods of proof of the global theorem do not generalize to cone-manifolds. In the original proof one uses the completeness of the hyperbolic structure to convert the problem to that of studying the action of a discrete group of Möbius transformations on the Riemann sphere, $\widehat{\mathbb{C}}$. For a cone-manifold the group will not be discrete and the relationship between the action of the group on $\widehat{\mathbb{C}}$ and the hyperbolic structure is unclear. In our proof of Theorem 1.1, little use is made of completeness nor is the action of the group on $\widehat{\mathbb{C}}$ studied. In particular, two results at the heart of the classical proof, the measurable Riemann mapping theorem and the zero area theorem for limit sets, are not used.

These methods allows us to extend Theorem 1.1 to cone-manifolds.

THEOREM 5.8. *If M is a geometrically finite cone-manifold without rank one cusps and all cone angles are $\leq 2\pi$ then M is locally rigid rel cone angles and the conformal boundary.*

We remark that Theorem 5.8 should still hold for structures with rank one cusps. In particular the classical proof of Theorem 1.1 does allow such cusps.

McMullen [Mc] has shown that local rigidity of geometrically finite cone-manifolds for cone angles *greater* than 2π implies the grafting conjecture for simple closed curves. This was one the original motivations of this work. Scannell and Wolf [SW] have recently proved this conjecture for all laminations using harmonic maps. Local rigidity for cone angles greater than 2π is still an open question.

The results in this paper also have applications to classical conjectures about Kleinian groups. See [Br2], [Br1], [BB] and [BBES].

We now outline the contents of the paper.

The main object of study is the bundle, $E \rightarrow M$, of germs of Killing fields over the hyperbolic manifold M . Weil [We2] showed that the deRham cohomology group $H^1(M; E)$ is canonically isomorphic to the Zariski tangent space of $R(M)$, the space of representations of $\pi_1(M)$ in $\mathrm{PSL}_2\mathbb{C}$ modulo conjugacy. E has a flat connection which gives a covariant derivative, d , and a natural Riemannian metric which allows us to define a co-derivative, δ , and a Laplacian, Δ .

If M is closed manifold then the Hodge theorem implies that every cohomology class in $H^1(M; E)$ has a harmonic representative. One then shows via a Weitzenböck formula that any harmonic representative is trivial. If the manifold is not closed then

we no longer have the Hodge theorem and a boundary term appears in the Weitzenbock formula. In this case we prove a Hodge theorem for every cohomology class that has a representative, ω , that is ‘almost’ harmonic, in the sense that $\delta\omega$ is in L^2 . Our first step is to construct a model deformation that is ‘almost’ harmonic to which we apply the Hodge theorem. We then find an exhaustion of the manifold by compact submanifolds and apply the Weitzenbock formula to the harmonic representative restricted to these compact submanifolds. The last step is to show the boundary term in the Weitzenbock term approaches zero as we exhaust the manifold. This will only happen if the cohomology class represents a deformation that fixes the cone angles and the conformal boundary.

In Section 2 we summarize the necessary background contained in Sections 1, 2 of [HK]. We emphasize those calculations which will be used later in this paper.

Section 3 is the heart of the paper. In it we construct the model deformation on a geometrically finite end.

In Section 4 we prove the Hodge theorem for the model deformation. We use the Hodge theorem to prove a vanishing theorem for those cohomology classes that fix the cone angle and the conformal boundary.

In the conclusion of the paper, Section 5, we make the identification of $H^1(M; E)$ with the Zariski tangent space of $R(M)$. Using our calculation of $H^1(M; E)$ we show that $R(M)$ is locally parameterized by the Teichmüller space of the conformal boundary and the complex length of the cone singularity. This then implies the main result, Theorem 5.8.

2. Background Deformation Theory

Let M be a manifold and $\rho: \pi_1(M) \rightarrow \mathrm{PSL}_2\mathbb{C}$ a representation of its fundamental group. Let $\tilde{E}(M) = \tilde{M} \times \mathrm{sl}_2\mathbb{C}$ and let $E(M)$ be the quotient $\tilde{E}(M)/\pi_1(M)$ where $\pi_1(M)$ acts on the first factor as covering transformations and via the adjoint representation on the second factor.

We will be most interested in two cases: when ρ is the holonomy representation of a hyperbolic structure on a 3-manifold M or a projective structure on a surface S . In the former case $E(M)$ is the bundle of germs of Killing fields on M and for the latter case $E(S)$ is the bundle of germs of projective vector fields on S .

The bundle $E(M)$ has a flat connection, d , which allows us to define deRham cohomology groups. Our main goal of this paper is to calculate $H^1(M; E(M))$ for a certain class of hyperbolic structures. In Section 5 we will see that $H^1(M; E(M))$ is the tangent space of the space of hyperbolic cone-structures on M .

For the remainder of this section we restrict to the case where $E = E(M)$ is the bundle of a hyperbolic 3-manifold M .

A hyperbolic structure on M can be defined by a *developing map*, $D: \tilde{M} \rightarrow \mathbb{H}^3$, and a *holonomy representation*, $\rho: \pi_1(M) \rightarrow \mathrm{Isom}^+\mathbb{H}^3$, where D is a local homeomorphism that commutes with the action of $\pi_1(M)$, i.e. $D(\gamma(x)) = \rho(\gamma)D(x)$ for all $\gamma \in \pi_1(M)$ and $x \in \tilde{M}$.

A smooth 1-parameter family of hyperbolic structures, M_ℓ , on M will have a smooth 1-parameter family of developing maps, D_t . We call such a family a *local deformation* of the hyperbolic structure. For each $x \in \tilde{M}$, $D_t(x)$ is a smooth path in \mathbb{H}^3 . The tangent vector at $D_0(x)$ can be pulled back by D_0 to a tangent vector at x defining a vector field v on \tilde{M} . This vector field represents an *infinitesimal deformation* of the hyperbolic structure. Differentiating

$$D_t(\gamma(x)) = \rho_t(\gamma)D_t(x)$$

we see that

$$v - \gamma_*v = \dot{\rho}(\gamma), \tag{2.1}$$

where $\dot{\rho}(\gamma)$ is the tangent vector of the path $\rho_t(\gamma)$ at $t = 0$. The tangent vector of a path in $\mathrm{PSL}_2\mathbb{C}$ is an element of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ which is implicitly identified with a Killing field in (2.1). The Killing field $\dot{\rho}$ describes the *infinitesimal change in holonomy* of γ induced by the deformation. In general if v is a vector field on \tilde{M} with $v - \gamma_*v$ a Killing field for all $\gamma \in \pi_1(M)$ then v is *automorphic*.

A vector field on M (or \tilde{M}) can be lifted to a section of E (or \tilde{E}). In fact this can be done in three different ways which we now describe. To do this we note that the Lie algebra, $\mathfrak{sl}_2\mathbb{C}$, has a complex structure that can be geometrically interpreted using the curl operator on vector fields. If $v \in \mathfrak{sl}_2\mathbb{C}$ is a Killing vector field on \mathbb{H}^3 then $\mathrm{curl} v$ will also be a Killing field and $\mathrm{curl} \mathrm{curl} v = -v$. (The curl we are using differs from the usual curl by a factor of $-1/2$). We'll say more about this below.) We can then define $iv = \mathrm{curl} v$. This will coincide with the usual complex structure on $\mathfrak{sl}_2\mathbb{C}$.

If s is a section of E and $p \in M$ then $s(p)$ is a Killing field in a neighborhood of p and $s(p)(p)$ will be a vector in the tangent space of M at p . We then define vector fields $\mathrm{Re} s$ and $\mathrm{Im} s$ by $\mathrm{Re} s(p) = s(p)(p)$ and $\mathrm{Im} s(p) = -(\mathrm{curl} s(p))(p)$. Since the Killing field $s(p)$ is uniquely determined by its value $s(p)(p)$ at p , and the value $(\mathrm{curl} s(p))(p)$ of its curl at p this defines an isomorphism between E and $TM \oplus TM$.

If v is a vector field on M we lift v to sections V and ιV of E by setting $\mathrm{Re} V = v$ and $\mathrm{Im} V = 0$ while $\mathrm{Re} \iota V = 0$ and $\mathrm{Im} \iota V = v$. Throughout the paper we will use this notational convention of denoting vector fields on M by lowercase letters and their corresponding sections of E by uppercase letters.

We now define one final method for lifting a vector field v to a section of E . For each point p on M we can find a Killing field, v_p , in a neighborhood of p that best approximates v at p . That is v_p will be the unique Killing field such that $v_p(p) = v(p)$ and $(\mathrm{curl} v_p)(p) = (\mathrm{curl} v)(p)$. The *canonical lift* of v , s , is defined by $s(p) = v_p$. Working through the definitions we see that $(\mathrm{curl} s(p))(p) = is(p)(p) = (\mathrm{curl} v)(p)$ so $\mathrm{Re} is = -\mathrm{Im} s = \mathrm{curl} v$ and $s = V - \iota \mathrm{curl} V$. (In [HK] this is the definition of the canonical lift.)

We define a section s of \tilde{E} to be *automorphic* if $s - \gamma_*s$ is constant. An automorphic vector field and an automorphic section both describe infinitesimal deformations of the hyperbolic structure on M . We have the following relationship.

PROPOSITION 2.1. *Let s be the canonical lift of an automorphic vector field v . Then s is an automorphic section.*

Proof. By naturality $s - \gamma_* s$ will be the canonical lift of $v - \gamma_* v$. Since v is automorphic, $v - \gamma_* v$ will be a Killing field. By definition the canonical lift of a Killing field will be constant. Therefore $s - \gamma_* s$ is constant and s is automorphic. \square

There are two simple but trivial ways to construct a local deformation of a hyperbolic structure. First we can post-compose the developing map with a smooth family σ_ℓ of isometries of \mathbb{H}^3 with $\sigma_0 = \text{id}$. In this case the associated automorphic vector field will be a Killing field. The other method is to precompose the developing map with the lift of an isotopy of M . In this case the associated automorphic vector field will actually be an equivariant vector field. For this reason we say that an infinitesimal deformation is trivial if it is the sum of a Killing field and an equivariant vector field. In terms of sections a deformation is trivial if it is a constant section plus an equivariant section. Two deformations are equivalent if they differ by a trivial deformation. This definition holds for both vector fields and sections.

If s is an automorphic section then $\omega = ds$ will be an equivariant 1-form because

$$\omega - \gamma_* \omega = ds - \gamma_* ds = d(s - \gamma_* s) = 0$$

since $s - \gamma_* s$ is constant. Therefore ω descends to an E -valued 1-form on M . If s is equivariant then ω will be an exact 1-form. If s is constant ω will be zero. Also s_1 and s_2 will be equivalent deformations if and only if $\omega_1 = ds_1$ and $\omega_2 = ds_2$ differ by an exact 1-form. Therefore the deRham cohomology group $H^1(M; E)$ describes the space of infinitesimal deformations.

Remark. For a vector field v on \tilde{M} with canonical lift s the E -valued 1-form, $\omega = ds$ should be compared with Thurston's description of the Schwarzian derivative of complex analysis. In particular, if f is a univalent holomorphic function then for each z we can find an *osculating Möbius transformation*, M_z^f , which is the unique Möbius transformation whose 2-jet agrees with the 2-jet of f at z . The *Schwarzian derivative* of f is the derivative of M_z^f in $\text{PSL}_2\mathbb{C}$. This definition makes it apparent that the Schwarzian measures how far f differs from a projective map just as ω measure how far v differs from a Killing field.

We now define a metric on E and \tilde{E} . If $x \in \mathbb{H}^3$ and $v, w \in \mathfrak{sl}_2\mathbb{C}$ we can define an inner product on $\mathfrak{sl}_2\mathbb{C}$ depending on x by

$$\langle v, w \rangle_x = \langle v(x), w(x) \rangle + \langle iv(x), iw(x) \rangle$$

where \langle, \rangle is the standard inner product of \mathbb{H}^3 . If $\gamma \in \text{PSL}_2\mathbb{C}$ then

$$\langle v, w \rangle_x = \langle \gamma_* v, \gamma_* w \rangle_{\gamma(x)} \tag{2.2}$$

where γ_* acts on $\mathfrak{sl}_2\mathbb{C}$ by the adjoint representation. Via the developing map this defines an inner product on the fibers of \tilde{E} . By (2.2) this inner product is invariant under the action of $\pi_1(M)$ and therefore descends to an inner product on the fibers of E .

The inner product determines a bundle map from E to the dual bundle E^* . If α is an E -valued form we write its E^* -valued dual as α^\sharp . For an E^* -valued form, α , the dual form is α^\flat . In local coordinates we can write any k -form as a sum of terms of the form $s\omega$ with s an E -valued section and ω a real valued k -form. We then use the Hodge $*$ -operator for the hyperbolic metric on real forms to define $*(s\omega) = s(*\omega)$ and $*(s\omega)^\sharp = (s^\sharp)(* \omega)$ and extend the definition to a arbitrary E -valued k -form linearly. It is easy to see that this local definition is well defined and this allows us to define an inner product on E -valued k -forms α and β by

$$(\alpha, \beta) = \int_M \alpha \wedge * \beta^\sharp. \quad (2.3)$$

Here the wedge product between an E -valued form and an E^* -valued form is a real form. We also define

$$\|\alpha\|^2 = \alpha \wedge * \alpha^\sharp.$$

The bundle E^* will also have a flat connection with exterior derivative d^* . However, differentiating in E^* is not the same as differentiating in E . More explicitly let $\partial\omega = (d * \omega^\sharp)^\flat$. We shall see shortly that $d \neq \partial$. We use ∂ to define a formal adjoint for d . Let $\delta = (-1)^k * \partial *$ where δ acts on an E -valued k -form on a three-dimensional hyperbolic manifold. Then $(d\alpha, \beta) = (\alpha, \delta\beta)$ if α and β are C^∞ k -forms with compact support, i.e. δ is the formal adjoint of d . We can now define the Laplacian, $\Delta = d\delta + \delta d$.

In local coordinates there is a nice formula for d and δ in terms of the Riemannian connection ∇ and algebraic operators. If $\{e_i\}$ is an orthonormal frame field with dual co-frame field $\{\omega^i\}$ we have

$$d = \sum_i \omega^i \wedge (\nabla_{e_i} + \text{ad}(E_i)) \quad (2.4)$$

and

$$\delta = - \sum_j i(e_j)(\nabla_{e_j} - \text{ad}(E_j)). \quad (2.5)$$

Here $i()$ is the interior product on forms. The operator $\text{ad}(E_i)$ takes a Killing field Y to the Killing field $[E_i, Y]$ where $[\cdot, \cdot]$ is the usual bracket on vector fields. (Recall that E_i is the real lift of e_i .) We also need to decompose d and δ into their real and symmetric parts. That is let $D = \text{Re } d$, $T = \text{Im } d$, $D^* = \text{Re } \delta$ and $T^* = \text{Im } \delta$. Note that D^* and T^* are the formal adjoints of D and T respectively. It is also worth noting that $\partial = D - T$.

In [MM] it is calculated that $T^*D + D^*T + TD^* + DT^* = 0$ and therefore Δ is a real operator. This leads to a Weitzenböck formula

$$\Delta = \Delta_D + H \quad (2.6)$$

where $\Delta_D = D^*D + DD^*$ and $H = T^*T + TT^*$. Note that H is a purely algebraic operator.

The tangent bundle, TM , has an exterior derivative \hat{d} and the hyperbolic metric gives a co-derivative, $\hat{\delta}$. For a vector field v let \hat{v} be the dual 1-form. We use similar notation to define the real valued Laplacian. Namely let $\hat{\Delta} = \hat{d}\hat{\delta} + \hat{\delta}\hat{d}$. There is also a Weitzenböck formula relating the two Laplacians:

$$\widehat{\Delta V} = \widehat{\Delta v} + 4\hat{v}. \quad (2.7)$$

The divergence and curl of v can be defined in terms of \hat{d} and $\hat{\delta}$. That is, $\operatorname{div} v = \hat{\delta}\hat{v}$ is the *divergence* of v and the *curl* of v is defined by the formula $\widehat{\operatorname{curl}} v = -(1/2) * \hat{d}\hat{v}$. (One usually defines the curl to be the vector field dual to $*\hat{d}\hat{v}$. The factor of $-(1/2)$ is chosen such the curl of a Killing field is that same as multiplication by ι in $\mathfrak{sl}_2\mathbb{C}$.)

For a vector field v , ∇v is a tensor $(1,1)$, i.e. a section of the bundle $\operatorname{Hom}(TM, TM)$. The divergence, curl and a third quantity, the strain, completely determine ∇v . In particular, $\operatorname{div} v$ is the trace of ∇v and measures the infinitesimal change in volume. By definition the *strain*, $\operatorname{str} v$, of v is the symmetric, traceless part of ∇v . It measures the infinitesimal change in conformal structure. The divergence and strain together measure the infinitesimal change in metric. The skew-symmetric part of ∇v , $\operatorname{skew} \nabla v$, is naturally identified with curl of v . More explicitly there is an isomorphism from TM to skew-symmetric sections of $\operatorname{Hom}(TM, TM)$. We define this isomorphism by choosing an orthonormal frame field $\{e_1, e_2, e_3\}$ with dual co-frame field $\{\omega^1, \omega^2, \omega^3\}$ and sending e_i to $e_{i+1} \otimes \omega^{i+2} - e_{i+2} \otimes \omega^{i+1}$ (the indices are measured mod 3). Under this isomorphism $\operatorname{skew} \nabla v$ is exactly $\operatorname{curl} v$ as defined above. The Riemannian metric gives a norm to $\operatorname{Hom}(TM, TM)$ for which this decomposition is orthogonal.

The real and imaginary parts of an E -valued 1-form are both vector valued 1-forms or sections of $\operatorname{Hom}(TM, TM)$. Using the formulas above Hodgson and Kerckhoff relate the real and imaginary parts of an E -valued 1-form to the divergence, strain and curl of a vector field: These results, which we summarize in the following theorem, can be found in Section 2 of [HK].

THEOREM 2.2. *Let s be an automorphic section of \tilde{E} . Then there exists an automorphic vector field v and an equivariant vector field w such that $s = V - \iota \operatorname{curl} V + \iota W$. Moreover*

- (1) $\operatorname{sym} \operatorname{Re} ds = \nabla v$;
- (2) $\operatorname{skew} \operatorname{Re} ds = w$;
- (3) if v is divergence free and harmonic and $w \equiv 0$ then $\operatorname{Re} ds = \operatorname{str} v$ and $\operatorname{Im} ds = -\operatorname{str} \operatorname{curl} v$.

An E -valued 1-form ω is a *Hodge form* if there exists an automorphic, divergence free, harmonic vector field v with canonical lift s such that $\omega = ds$. There is a very simple formula for the L^2 -norm of a Hodge form. It is essentially Proposition 1.3 of [HK].

PROPOSITION 2.3. *Let M be a compact hyperbolic manifold with boundary and ω a Hodge form on M . Then*

$$\int_M \|\omega\|^2 = \frac{1}{2} \int_{\partial M} \iota\omega \wedge \omega^\sharp$$

where ∂M is oriented with inward pointing normal.

3. Geometrically Finite Ends

3.1. PROJECTIVE STRUCTURES AND GEOMETRICALLY FINITE ENDS

Throughout this section we let $M = S \times [0, \infty)$ where S is a closed surface of genus >1 . We also assume that M has a complete hyperbolic structure with boundary $S \times \{0\}$.

A *projective structure*, Σ , on a surface is given by an atlas of charts with image in $\widehat{\mathbb{C}}$ and transition maps Möbius or projective transformations. As with hyperbolic structures, a projective structure can be given by a developing map and a holonomy representation. If Σ_1 and Σ_2 are projective structures then $\Sigma_1 \cong \Sigma_2$ if there exists a projective homeomorphism from Σ_1 to Σ_2 .

DEFINITION 3.1. *M is a geometrically finite end without rank one cusps if it is compactified by a projective structure Σ on $S \times \{\infty\}$. Then Σ is the projective boundary of M .*

Since we will not discuss rank one cusps in this paper we will simply refer to such ends as geometrically finite.

To see this definition more explicitly we recall that $\widehat{\mathbb{C}}$ naturally compactifies \mathbb{H}^3 . We refer to this compactification as $\tilde{\mathbb{H}}^3 = \mathbb{H}^3 \cup \widehat{\mathbb{C}}$. Then $\mathrm{PSL}_2\mathbb{C}$ acts continuously on $\tilde{\mathbb{H}}^3$ as isometries of \mathbb{H}^3 and projective transformations of $\widehat{\mathbb{C}}$. Then M is geometrically finite if it has an atlas of hyperbolic charts that extends continuously to a atlas for a projective structure on $S \times \{\infty\}$. In fact if M is geometrically finite there will be a developing map

$$D: \tilde{S} \times [0, \infty] \longrightarrow \tilde{\mathbb{H}}^3$$

that restricts to a developing map for the hyperbolic structure M on $\tilde{S} \times [0, \infty)$ and a developing map for the projective structure Σ on $\tilde{S} \times \{\infty\}$.

We refer to the bundles $E(\Sigma)$ and $E(\tilde{\Sigma})$ as E_∞ and \tilde{E}_∞ , respectively. E_∞ and \tilde{E}_∞ are the bundles of germs of projective vector fields on Σ and $\tilde{\Sigma}$, respectively.

The product structure on $M \cup \Sigma$ allows us to define a projection map

$$\Pi: M \longrightarrow \Sigma$$

by the formula $\Pi(p, t) = p$. Let $\tilde{\Pi}$ be a lift of Π to the universal covers \tilde{M} and $\tilde{\Sigma}$. The corresponding bundles \tilde{E} and \tilde{E}_∞ have canonical product structures so we can use $\tilde{\Pi}$

to pull back sections of \tilde{E}_∞ to sections of \tilde{E} . If we restrict $\tilde{\Pi}$ to $\tilde{S} \times \{0\}$ in \tilde{M} we can also push forward sections of \tilde{E} to sections of \tilde{E}_∞ .

LEMMA 3.2. *Let s be an automorphic section of \tilde{E} and s_∞ an automorphic section of \tilde{E}_∞ .*

- (1) $\tilde{\Pi}_* s$ and $\tilde{\Pi}^* s_\infty$ will be equivariant iff s and s_∞ , respectively, are equivariant.
- (2) $\tilde{\Pi}_* s$ and $\tilde{\Pi}^* s_\infty$ are automorphic.
- (3) Automorphic sections s' and s'_∞ will be equivalent as infinitesimal deformations to s and s_∞ , respectively, iff $\tilde{\Pi}_* s'$ and $\tilde{\Pi}^* s'_\infty$ are equivalent to $\tilde{\Pi}_* s$ and $\tilde{\Pi}^* s_\infty$, respectively.

Therefore there are isomorphisms

$$\Pi_*: H^1(M; E) \longrightarrow H^1(S; E_\infty) \quad \text{and} \quad \Pi^*: H^1(S; E_\infty) \longrightarrow H^1(M; E).$$

Proof. The actions of $\pi_1(M) = \pi_1(S)$ on \tilde{E} and \tilde{E}_∞ , respectively, will commute with Π so

$$\tilde{\Pi}_* s - \gamma_* \tilde{\Pi}_* s = \tilde{\Pi}_*(s - \gamma_* s)$$

for all $\gamma \in \pi_1(M)$. This implies (1) for $\tilde{\Pi}_*$ and (1) then implies (2) and (3). The proof for $\tilde{\Pi}^*$ is similar.

Given a closed E -valued 1-form ω we can integrate ω to find an automorphic section, s , of \tilde{E} such that $ds = \omega$. We then define $\Pi_* \omega = d\Pi_* s$. We similarly define Π^* for an E_∞ -valued 1-form. (1), (2) and (3) imply that this defines maps between $H^1(M; E)$ and $H^1(S; E_\infty)$. If $s' = \Pi^* \Pi_* s$ then $s = s'$ on $\tilde{S} \times \{0\}$ so ds' is cohomologous to ds . Therefore the map Π_* is one-to-one from $H^1(M; E)$ to $H^1(S; E_\infty)$ and by similar reasoning Π^* is one-to-one from $H^1(S; E_\infty)$ to $H^1(M; E)$ and both $\Pi^* \Pi_*$ and $\Pi_* \Pi^*$ are the identity map. Therefore the maps Π_* and Π^* are isomorphisms. \square

Remark. In general a vector field on a geometrically finite end will not extend continuously to the conformal boundary. What Lemma 3.2 allows us to do is replace an automorphic vector field, v , on \tilde{M} with an equivalent vector field that does extend continuously. Namely if s is the canonical lift of v then $v' = \text{Re } \Pi^* \Pi_* s$ will be equivalent to v and $v' \cup \text{Re } \Pi_* s$ will be continuous on $\tilde{M} \cup \tilde{S}$.

3.2. EXTENDING SECTIONS VIA HOROSPHERE PROJECTIONS

We will need to extend automorphic vector fields on $\tilde{\Sigma}$ to harmonic, divergence free vector fields on \tilde{M} . We first describe a method for extending a vector field on \mathbb{C} to \mathbb{H}^3 . More precisely we extend a vector field to a harmonic section of $E(\mathbb{H}^3)$.

We will use the following orthonormal frame field on \mathbb{H}^3 : Working in the upper half space model, $\mathbb{H}^3 = \{(x, y, t) : t > 0\}$, let $e_1 = t(\partial/\partial x)$, $e_2 = t(\partial/\partial y)$ and

$e_3 = t(\partial/\partial t)$ with corresponding real E -valued sections E_i . Also let $R_i = tE_i$ and let ω^i be the dual real 1-form for e_i . Using (2.4) and (2.5) we see that

$$\begin{aligned} dE_1 &= E_3\omega^1 + R_3\omega^2 - R_2\omega^3 \\ dE_2 &= -R_3\omega^1 + E_3\omega^2 + R_1\omega^3 \\ dE_3 &= -(E_1 - R_2)\omega^1 - (R_1 + E_2)\omega^2 \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} \partial E_1 &= E_3\omega^1 - R_3\omega^2 + R_2\omega^3 \\ \partial E_2 &= R_3\omega^1 + E_3\omega^2 - R_1\omega^3 \\ \partial E_3 &= -(E_1 + R_2)\omega^1 + (R_1 - E_2)\omega^2. \end{aligned} \quad (3.9)$$

If v is a projective vector field on $\mathbb{C} \subset \widehat{\mathbb{C}}$ then there is an obvious way to extend v to a harmonic vector field on \mathbb{H}^3 . Namely there is a unique Killing field on \mathbb{H}^3 that extends continuously to v on \mathbb{C} . Every projective vector field is of the form $v(z) = p(z)(\partial/\partial z)$ where p is a quadratic polynomial. Then the section

$$\frac{p(w)}{t}(E_1 - R_2) + p_z(w)E_3 - \frac{tp_{zz}(w)}{2}(E_1 + R_2) \quad (3.10)$$

is constant and evaluates at every point to the Killing field that extends continuously to v .

Now let $v(z) = f(z)(\partial/\partial z)$ be an arbitrary smooth vector field on \mathbb{C} . Then the *canonical lift*, $s_\infty(z)$, of v is the section of $E(\mathbb{C})$ defined by the formula

$$s_\infty(w) = \left(f(w) + f_z(w)(z - w) + \frac{f_{zz}(w)}{2}(z - w)^2 \right) \frac{\partial}{\partial z}. \quad (3.11)$$

Similar to the canonical lift of vector fields on hyperbolic space, $s_\infty(w)$ is the projective vector field that best approximates v at w . Next we define a section of $E(\mathbb{H}^3)$ by the formula $s(w, t) = s_\infty(w)$. Using (3.10) we see that

$$s(w, t) = \frac{f(w)}{t}(E_1 - R_2) + f_z(w)E_3 - \frac{tf_{zz}(w)}{2}(E_1 + R_2). \quad (3.12)$$

Note that the vector fields $\operatorname{Re} s$ and $\operatorname{Im} s$ extend continuously to v and $-w$, respectively.

To calculate Δs we use the formula,

$$\Delta(fs) = (\widehat{\Delta}f)s - 2 * (*\hat{d}f \wedge Ds) + f(\Delta s). \quad (3.13)$$

where f is a complex valued function and s an E -valued section. If f is a function in the upper half space model of \mathbb{H}^3 then

$$\hat{d}f = tf_x\omega^1 + tf_y\omega^2 + tf_t\omega^3 \quad (3.14)$$

and

$$\widehat{\Delta}f = tf_t - t^2(f_{xx} + f_{yy} + f_{tt}). \quad (3.15)$$

After a straightforward but long calculation, using these formulas, we have:

$$\Delta s(w, t) = -2t f_{z\bar{z}}(w)(E_1 - R_2) - 2t^2 f_{z\bar{z}\bar{z}}(w)E_3 + 2t^3 f_{z\bar{z}\bar{z}\bar{z}}(w)(E_1 + R_2). \quad (3.16)$$

There are several things to notice in this formula. If $v = f(\partial/\partial z)$ is conformal, i.e. f is a holomorphic function, then $\Delta s = 0$. If v is not conformal then $\|\Delta s\| \rightarrow 0$ as $t \rightarrow 0$. In other words as we approach the ideal boundary the norm of Δs goes to zero. In fact we can estimate the rate of decrease. Let $d(w, t) = -\log t$ be the distance of a point from the horosphere $t = 1$. Then $\|\Delta s(w, t)\| = 4|f_{z\bar{z}}(z)|e^{-d(w,t)} + o(t^2)$. ($o(t)$ is a function such that $(o(t)/t)$ is bounded as $t \rightarrow 0$.)

For a conformal vector field there is also a nice expression for ds . Namely

$$ds = -\frac{t^2 f_{z\bar{z}\bar{z}}}{2}(E_1 + R_2)(\omega^1 + t\omega^2). \quad (3.17)$$

Here we note that $\operatorname{Re} ds$ and $\operatorname{Im} ds$ are symmetric and traceless, therefore they are the strains of the divergence free vector fields $\operatorname{Re} s$ and $\operatorname{Im} s$, respectively. We can also calculate the norm, $\|ds(p, t)\| = |f_{z\bar{z}}(w)|e^{-2d}$, which should be thought of as the norm of the strains of $\operatorname{Re} s$ and $\operatorname{Im} s$.

Remark. By differentiating (3.11) we see that

$$ds_\infty(w) = \left(\frac{f_{z\bar{z}\bar{z}}(w)}{2}(z-w)^2 \frac{\partial}{\partial z} \right) dw.$$

The quantity $f_{z\bar{z}\bar{z}}(w)$ is the infinitesimal version of the Schwarzian derivative discussed in the remark in Section 2. In particular we see that the Schwarzian derivative of the vector field at infinity determines the strain of the extended vector field in hyperbolic space.

3.3. CONVEX PARALLEL SURFACES IN \mathbb{H}^3

We now describe some results on parallel surfaces in hyperbolic space. A detailed study of such surfaces can be found in [Ep]. All of the results described in this section can be found in Section 3 of this paper.

Let S be a smooth convex surface in \mathbb{H}^3 . For a point $p \in S$ normalize the position of S in \mathbb{H}^3 such that $p = (0, 0, 1)$ in the upper half space model and such that the directions of principal curvature at p are $(1, 0, 0)$ and $(0, 1, 0)$. In a neighborhood V of the geodesic ray normal to S at p we can foliate \mathbb{H}^3 by surfaces S_t such that each S_t is equidistant from S and contains the point $(0, 0, t)$. Note that the distance between S and S_t is $-\log t$. There are maps $\pi_t: S \rightarrow S_t$ that take each point in S to the nearest point in S_t . There is also a projection $\Pi: V \rightarrow \mathbb{C}$ that takes each point on S_t to the limit of its normal ray in \mathbb{C} . Note that Π restricted to S is the limit of the maps π_t as $t \rightarrow 0$.

Let $k_1(t)$ and $k_2(t)$ be the principal curvatures of the surface S_t at the point $(0, t)$. Then

$$k_i(t) = \frac{1 + k_i(0) + t^2(k_i(0) - 1)}{1 + k_i(0) + t^2(1 - k_i(0))}. \quad (3.18)$$

We also need to calculate the derivatives of the π_t and Π . We will express the answer in terms of the orthonormal frame field $\{e_1, e_2, e_3\}$ for \mathbb{H}^3 defined in the previous section and the frame field $\{(\partial/\partial x), (\partial/\partial y)\}$ for \mathbb{C} . Then the derivative of π_t at $(0, 0, 1)$ is

$$(\pi_t)_* = \begin{pmatrix} \frac{1 + k_1(0) + t^2(1 - k_1(0))}{2t} & 0 \\ 0 & \frac{1 + k_2(0) + t^2(1 - k_2(0))}{2t} \end{pmatrix} \quad (3.19)$$

and the derivative of Π at $(0, 0, t)$ is

$$\Pi_* = \begin{pmatrix} \frac{(1 + k_1(t))t}{2} & 0 & 0 \\ 0 & \frac{(1 + k_2(t))t}{2} & 0 \end{pmatrix} \quad (3.20)$$

3.4. EXTENDING SECTIONS VIA CONVEX SURFACES

We continue to assume that S be an embedded convex surface in \mathbb{H}^3 with position normalized as in the previous section. However, in this section it will be convenient to use complex coordinates for the upper half space model. That is we view the first two real coordinates as a single complex coordinate.

Let v be a conformal vector field on \mathbb{C} let s_∞ be the canonical lift of v . Define a section of $E(\mathbb{H}^3)$ by the formula $s = \Pi^* s_\infty$. We will calculate ds and Δs along the ray $(0, t)$.

PROPOSITION 3.3. $\|ds(0, t)\|, \|(\Delta s)(0, t)\|, \|(\operatorname{div} \operatorname{Re} s)(0, t)\|$ and $\|\hat{d}(\operatorname{div} \operatorname{Re} s)(0, t)\| = o(t^2)$.

Proof. For the special case when S is a horosphere we calculated s , ds and Δs in Section 3.2. In general s will be the sum of the horosphere extension, s_h , and a correction term s_c .

Let $z = \Pi(w, t)$. Using (3.10) we see

$$\begin{aligned} s(w, t) &= \left[f(z) + f_z(z)(w - z) + \frac{f_{zz}(z)(w - z)^2}{2} \right] \frac{E_1 - R_2}{t} + \\ &\quad + [f_z(z) + f_{zz}(z)(w - z)]E_3 - f_{zz}(z) \frac{t(E_1 + R_2)}{2} \\ &= [f(w) - g_1(w, z)] \frac{E_1 - R_2}{t} + \\ &\quad + [f_z(w) - g_z(w, z)]E_3 - \\ &\quad - [f_{zz}(w) - g_3(w, z)] \frac{t(E_1 + R_2)}{2} \end{aligned}$$

where

$$\begin{aligned} g_1(w, z) &= (w - z)^3 \sum_{n=3}^{\infty} \frac{f^{(n)}(z)}{n!} (w - z)^{n-3} \\ g_2(w, z) &= (w - z)^2 \sum_{n=2}^{\infty} \frac{f^{(n+1)}(z)}{n!} (w - z)^{n-2} \\ g_3(w, z) &= (w - z) \sum_{n=1}^{\infty} \frac{f^{(n+2)}(z)}{n!} (w - z)^{n-1}. \end{aligned} \quad (3.21)$$

Therefore $s = s_h - s_c$ where,

$$s_h(w, t) = \frac{f(w)}{t}(E_1 - R_2) + f_z(w)E_3 - \frac{tf_{zz}(w)}{2}(E_1 + R_2), \quad (3.22)$$

and,

$$s_c(w, t) = \frac{g_1(w, z)}{t}(E_1 - R_2) + g_2(w, z)E_3 - \frac{tg_3(w, z)}{2}(E_1 + R_2). \quad (3.23)$$

We have already calculated ds_h and Δs_h so we left to calculate ds_c and Δs_c .

Let $G_i(w, t) = g_i(w, \Pi(w, t))$ for $i = 1, 2, 3$. By our normalization $\Pi(0, t) = 0$ for all t , so $w - z = 0$ when $w = 0$. Furthermore the G_i extend smoothly to \mathbb{C} with $G_i(w, 0) = g_i(w, \Pi(w, 0)) = g_i(w, w) = 0$. For this reason the Euclidean derivatives of the G_i will be bounded on $(0, t)$. Then (3.14) and (3.15) imply that $\|\hat{d}G_i(0, t)\| = o(t)$ and $\|\hat{\Delta}G_i(0, t)\| = o(t^2)$.

In fact using the product formula for the real Laplacian,

$$\hat{\Delta}(fg) = (\hat{\Delta}f)g - 2(\hat{d}f, \hat{d}g) + f(\hat{\Delta}g), \quad (3.24)$$

and the Leibniz rule we obtain

$$\begin{aligned} G_1(0, t) &= \hat{d}G_1(0, t) = \hat{\Delta}G_1(0, t) = 0 \\ G_2(0, t) &= \hat{d}G_2(0, t) = 0 \\ G_3(0, t) &= 0. \end{aligned} \quad (3.25)$$

(3.25) implies that

$$ds_c(0, t) = -\frac{t}{2}(E_1 + R_2)\hat{d}G_3(0, t) \quad (3.26)$$

so $\|ds_c(0, t)\| = o(t^2)$. We saw in (3.17) that $\|ds_h(0, t)\| = o(t^2)$ so $\|ds(0, t)\| = o(t^2)$. Furthermore $\operatorname{div} \operatorname{Re} s$ is the trace of $\operatorname{Re} ds$, so $\|(\operatorname{div} \operatorname{Re} s)(0, t)\| = o(t)^2$.

To estimate $\hat{d}(\operatorname{div} \operatorname{Re} s)$ we will need to know $\operatorname{div} \operatorname{Re} s$ more explicitly. Here we use (3.20) from the previous section to find

$$\hat{d}G_3(0, t) = \frac{tf_{zzz}(0)}{2}((1 - k_1(t))w^1 + t(1 - k_2(t))w^2). \quad (3.27)$$

Therefore

$$ds_c(0, t) = -\frac{t^2 f_{zzz}(0)}{4}(E_1 + R_2)((1 - k_1(t))w^1 + i(1 - k_2(t))w^2)$$

and

$$(\operatorname{div} \operatorname{Re} s_c)(0, t) = \frac{t^2 \operatorname{Re} f_{zzz}(0)}{4}(k_1(t) - k_2(t)). \tag{3.28}$$

Since $\operatorname{div} \operatorname{Re} s_h = 0$, $\operatorname{div} \operatorname{Re} s = \operatorname{div} \operatorname{Re} s_c$ and therefore by (3.28) $(\operatorname{div} \operatorname{Re} s)/t$ will extend to a smooth function \mathbb{C} in a neighborhood of zero. We again apply (3.14) to see that $\|\hat{d}(\operatorname{div} \operatorname{Re} s)(0, t)\| = o(t)^2$.

We are now left to calculate $(\Delta s_c)(0, t)$. By (3.13) and (3.25)

$$\begin{aligned} (\Delta s_c)(0, t) &= (\widehat{\Delta} G_2(0, t))E_3 + (\widehat{\Delta} G_3(0, t))\frac{t(E_1 + R_2)}{2} \\ &\quad - 2 * \left(* \hat{d} G_3(0, t) \wedge D\left(\frac{t(E_1 + R_2)}{2}\right) \right). \end{aligned} \tag{3.29}$$

We have estimated every term on the right except $D(t(E_1 + R_2)/2)$ which can calculate using (3.8) to see that it is $o(t)$. Therefore $\|(\Delta s_c)(0, t)\| = o(t^2)$ and since $\Delta s_h = 0$, $\|(\Delta s)(0, t)\| = o(t^2)$. □

3.5. EXTENDING VECTOR FIELDS ON THE BOUNDARY OF A GEOMETRICALLY FINITE END

We now return to our geometrically finite end, M . To apply the results of the previous two sections we need the following theorem:

THEOREM 3.4. *If M is a geometrically finite end then there is a smooth, embedded, convex surface in M whose inclusion is a homotopy equivalence.*

Proof. Let Σ be the projective boundary of M . For every projective structure there is a conformal developing map $f: U \rightarrow \widehat{\mathbb{C}}$ where U is the upper half plane in \mathbb{C} . We will extend f to a developing map for M .

Let M_z^f be the unique Möbius transformation whose 2-jet agrees with f at z . Let P be the hyperbolic plane whose boundary is the real line on $\widehat{\mathbb{C}}$. For each $p \in \mathbb{H}^3$ there is a unique geodesic g through p which is orthogonal to P . The geodesic g will have exactly one endpoint $z \in U$. Define a map $F: \mathbb{H}^3 \rightarrow \mathbb{H}^3$ by the formula $F(p) = M_z^f(p)$. The construction is natural so F will be a developing map for a geometrically finite end if F is a local diffeomorphism. Furthermore F extends continuously to f so this end will have projective boundary Σ .

The upper half plane U and the hyperbolic plane P bound a half space H in \mathbb{H}^3 . Let H_d be the set of points in H whose distance from P is $\geq d$. In Section 3 of [And] the derivative of F is calculated. In particular it is shown that for d sufficiently large F is a local diffeomorphism restricted to H_d (See p. 35). By Proposition 3.13 and

Theorem 3.17, $F(\partial H_d)$ will be convex for d sufficiently large. For such d , F restricted to H_d will be a developing map for a geometrically finite end M_d with projective boundary Σ . The boundary of M_d will be a convex surface.

The isomorphism from the projective boundary of M_d and the projective boundary of M will extend to an isometry from M_d to M . Although this isometry may not be defined on all of M_d it can be defined on all but a compact submanifold of M_d . Therefore we can choose d even larger so that this isometry is defined on all M_d . The image of ∂M_d under this isometry will be the desired convex surface. \square

The convex surface we have constructed separates M into a compact piece and a noncompact piece. The outward normal of the convex surface points into the noncompact piece so if we remove the compact piece M will have concave boundary. From now on we will assume this is the case. That is $\partial M = S \times \{0\}$ is concave. We also assume that the product structure is chosen such that for a fixed $p \in S$ the path $\{p\} \times [0, \infty)$ is a geodesic ray in M normal to $S \times \{0\}$. Furthermore, we assume that the second parameter is a unit speed parameterization of the geodesic. This implies that $S \times \{t\}$ is convex for all $t > 0$.

If v is an automorphic vector field on $\tilde{\Sigma}$ we can define a canonical lift of v on a local projective chart as in (3.11). One can then check to see that this lift is independent of the choice of chart so v has a well-defined canonical lift, s_∞ , to all of \tilde{S} . Furthermore if v is automorphic s_∞ will also be automorphic. Let $s = \Pi^* s_\infty$. Then Lemma 3.2 implies that s is also automorphic and therefore ds , Δs and $\operatorname{div} \operatorname{Re} s$ will be equivariant and descend to objects of the appropriate type on M .

THEOREM 3.5. *If v is a conformal vector field then ds , Δs , $\operatorname{div} \operatorname{Re} s$, and $\hat{d}(\operatorname{div} \operatorname{Re} s)$ all have finite L^2 norm on M .*

Proof. Let $p \in M$ lie on $S \times \{t\}$. Proposition 3.3 implies that there exists a continuous function $K: S \rightarrow \mathbb{R}^+$ such that $\|ds(p)\| < K(\Pi(p))e^{-2t}$. Since S is compact K will be bounded by some $K_1 > 0$.

Let dA_t be the area form for $S \times \{t\}$. Note that by (3.18) and (3.19), $\operatorname{area}(S \times \{t\}) < K_2 e^{2t} \operatorname{area}(S \times \{0\})$ where K_2 is determined by the maximal principal curvature on $S \times \{0\}$. Then

$$\begin{aligned} \int_M \|ds(p)\|^2 &= \int_0^\infty \int_{\Sigma_t} \|ds(p)\|^2 dA_t dt \\ &< \int_0^\infty (K_1)^2 e^{-4t} K_2 \operatorname{area}(S \times \{0\}) e^{2t} dt \\ &= \int_0^\infty (K_1)^2 K_2 \operatorname{area}(S \times \{0\}) e^{-2t} dt < \infty \end{aligned}$$

and ds is in L^2 .

The proof for the other terms is similar. \square

3.6. HARMONIC DEFORMATIONS OF RANK TWO CUSPS

A rank two cusp is the quotient of a horoball by a $\mathbb{Z} \oplus \mathbb{Z}$ group of parabolic isometries. Again it is convenient to work in the upper half space model where isometries can be defined by their action on $\widehat{\mathbb{C}}$. Since any two parabolics of \mathbb{H}^3 are conjugate we can explicitly describe any cusp as the quotient of a horoball based at infinity by parabolics $\gamma_1(z) = z + 1$ and $\gamma_2(z) = z + \tau$ with $\text{Im } \tau > 0$. The cusp M is homeomorphic to $T \times [0, \infty)$ where T is a torus. We can choose this product structure such that each $T \times \{t\}$ is the quotient of a horosphere and therefore has an induced Euclidean metric. The conformal class of these metrics will be constant for $T \times \{t\}$ and it is determined by the Teichmüller parameter τ .

We defined γ_1 and γ_2 by their action on \mathbb{C} . The quotient of this action is a projective structure Σ on the torus. To construct models for deformations of the rank two cusp we will first describe deformations of Σ .

Since the Euler characteristic of Σ is 0 Poincaré duality implies that the complex dimension of $H^1(\Sigma; E(\Sigma))$ is 2. An automorphic vector field on $\tilde{\Sigma}$ determines a cohomology class in $H^1(\Sigma; E(\Sigma))$. We claim that the automorphic vector fields $v_1 = (z - \bar{z}/2)(\partial/\partial x)$ and $v_2 = (z^3 - z/6)(\partial/\partial z)$ determine cohomology classes that are a basis for $H^1(\Sigma; E(\Sigma))$.

We first examine v_1 . Note that $\tilde{\Sigma} = \mathbb{C}$ where \mathbb{C} has the natural projective structure it inherits as a subset of $\widehat{\mathbb{C}}$. It is then easy to check that v_1 is automorphic. In particular $v_1 - (\gamma_1)_*v_1 = 0$ and $v_1 - (\gamma_2)_*v_1 = \text{Im } \tau(\partial/\partial z)$ are projective vector fields. If the cohomology class v_1 generates is trivial then v_1 is the sum of an equivariant vector field and a projective vector field. For this to be true there must be a projective vector field v with $v(0) = v(1) = 0$ and $v(\tau) = v(\tau + 1) = \text{Im } \tau$. Since no such v exist the cohomology class is non-trivial.

Similar reasoning applied to v_2 and any non-zero linear combination of v_1 and v_2 shows that v_2 also generates a non-trivial cohomology class and together v_1 and v_2 determine a basis of $H^1(\Sigma; E(\Sigma))$.

Just as in Section 3.2 we can extend v_1 and v_2 to sections of $E(\mathbb{H}^3)$. By (3.12), v_1 extends to

$$s_1(w, t) = \frac{w - \tilde{w}}{2t}(E_1 - R_2) + \frac{1}{2}E_3$$

and v_2 extends to

$$s_2(w, t) = \frac{w^3 - w}{6t}(E_1 - R_2) + \frac{3w^2 - 1}{6}E_3 - \frac{wt}{2}(E_1 + R_2).$$

Both s_1 and s_2 will be automorphic sections with respect to the action of γ_1 and γ_2 on \mathbb{H}^3 . Therefore

$$ds_1 = -\frac{1}{2}(E_1 - R_2)(\omega^1 - i\omega^2)$$

and

$$ds_2 = -\frac{t^2}{2}(E_1 + R_2)(\omega^1 + i\omega^2)$$

are equivariant and restrict to E -valued 1-forms on M . By Proposition 3.2 there is an isomorphism between $H^1(\Sigma; E(\Sigma))$ and $H^1(M; E)$ and therefore ds_1 and ds_2 are a basis for $H^1(M; E)$.

We have shown that every cohomology class in $H^1(M; E)$ has a representative of the form

$$\omega = -\frac{b_1}{2}(E_1 - R_2)(\omega^1 - i\omega^2) - \frac{b_2 t^2}{2}(E_1 + R_2)(\omega^1 + i\omega^2).$$

PROPOSITION 3.6. *The E -valued 1-form, ω , is closed, co-closed and traceless. Furthermore ω is in L^2 if and only if $b_2 = 0$ and if ω is in L^2 then*

$$\int_M \|\omega\|^2 = \frac{|b_1|^2}{2} \text{Area}(\partial M).$$

Proof. By construction ω is closed. From our explicit description of ω we see that it is traceless. By (3.16) we see that $\Delta s_1 = \delta ds_1 = 0$ and $\Delta s_2 = \delta ds_2 = 0$. Therefore ω is co-closed.

The pointwise norm of ω is $\|\omega(w, t)\|^2 = |b_1|^2 + t^4|b_2|^2$. The last two facts follow from integrating this norm over M . \square

We next describe the infinitesimal change in holonomy determined by ω . Again it is easier to work with the projective structure on the torus and then use the isomorphism between $H^1(\Sigma; E(\Sigma))$ and $H^1(M; E)$. Let γ_t be a smooth path in $\text{PSL}_2\mathbb{C}$ with $\gamma_0(z) = z + \beta$. Then the derivative of γ_t at $t = 0$ will be a projective vector field $(a_0 + a_1z + a_2z^2)(\partial/\partial z)$. A straightforward calculation shows that the derivative of the trace of γ_t at $t = 0$ is $-\beta a_2$. Therefore if v is a vector field on \mathbb{C} automorphic with respect to γ_0 then the infinitesimal change in trace is determined by the z^2 -coefficient of the projective vector field $v - (\gamma_0)_*v = (a_0 + a_1z + a_2z^2)(\partial/\partial z)$.

We now apply this to the vector fields v_1 and v_2 . For both γ_1 and γ_2 the z^2 -coefficient of $v_1 - (\gamma_1)_*v_1$ is zero. Therefore the infinitesimal change in trace determined by v_1 is zero. For v_2 the z^2 -coefficient of $v_2 - (\gamma_1)_*v_2$ is $(1/2)$ so the infinitesimal change in trace of γ_1 is $-(1/2)$. The z^2 -coefficient of $v_2 - (\gamma_2)_*v_2$ is $(\tau/2)$ so the infinitesimal change in trace of γ_2 is $-(\tau^2/2)$.

Although v_1 does not change the holonomy of either γ_1 or γ_2 there is an infinitesimal change in the projective structure. In particular there is an infinitesimal change in the conformal structure. Recall that the Teichmüller space of the torus can be identified with the upper half plane $U = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$. Any point $\tau \in U$ determines a parallelogram with vertices $0, 1, \tau$ and $\tau + 1$. By identifying opposite sides of this parallelogram we obtain a conformal structure on the torus. The affine vector

field v_1 fixes the side between 0 and 1 so the infinitesimal change in the Teichmüller parameter is given by $v_1(\tau) = \text{Im } \tau$. Note that the Teichmüller metric on the Teichmüller space of the torus is the hyperbolic metric. In this metric the length of the vector $v_1(\tau)$ is 1. In particular its length does not depend on τ .

4. Hodge Theory of Deformations

We are now ready to begin our analysis of geometrically finite hyperbolic cone-manifolds. We begin with some definitions. Let N be a compact 3-manifold with boundary and let \mathcal{C} be a collection of simple closed curves in the interior of N . Let M be the interior of $N - \mathcal{C}$. A singular metric g on $\text{int } N$ is a *hyperbolic cone-metric* if g is smooth metric of constant sectional curvature $\equiv -1$ on M while in neighborhood of a point $p \in \mathcal{C}$ the metric has the form

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2$$

with θ measured modulo some $\alpha > 0$. On each component of c of \mathcal{C} , α will be constant. Then α is the cone angle of the cone singularity at c . We further say that g is geometrically finite (without rank one cusps) if g extends to a projective structure on each non-toral component of ∂N .

The complement of any compact core of M will contain ends of three types: geometrically finite ends, rank two cusps and neighborhoods of the cone singularity. For each geometrically finite end we choose a smooth convex surface as given by Theorem 3.4 and we let S_0 be the union of these surfaces. We also choose a small horoball neighborhood for each rank two cusp such that the boundary of each is a collection of pairwise disjoint embedded Euclidean tori. We denote the union of these neighborhoods and their boundary, \mathcal{H}_0 and H_0 , respectively. Finally, we fix a small ϵ such that the ϵ -neighborhood of the singular locus, \mathcal{C}_ϵ , is a collection of disjoint solid tori in the interior of N , with boundary T_ϵ . Note that if any of S_0 , H_0 or T_ϵ intersect we can choose smaller neighborhoods of each end such that all three surfaces are disjoint.

Now let M_0 be the compact core of M bounded by S_0 , H_0 and T_ϵ . Let S_t and H_t be distance t surfaces from S_0 and H_0 , respectively, and T_t the boundary of the t -neighborhood of the singular locus. Then we define M_t to be the compact core of M bounded by S_t , H_t and $T_{\epsilon/(1+t)}$.

The geometrically finite ends each define a projective structure. We label the union of these projective structures, Σ , and the bundle of germs of Killing fields over Σ , E_∞ . The surfaces S_t define a map Π from the geometrically finite ends to Σ . By Lemma 3.2 $\Pi_*: H^1(M; E) \rightarrow H^1(\Sigma; E_\infty)$ is an isomorphism on homology. A cohomology class $[\omega_\infty] \in H^1(\Sigma; E_\infty)$ is *conformal* if there exists an automorphic, conformal vector field v on $\tilde{\Sigma}$ with canonical lift s such that $ds \in [\omega_\infty]$. A cohomology class $[\omega] \in H^1(M; E)$ is *conformal at infinity* if $\Pi_*[\omega]$ is conformal.

A cohomology class that is conformal at infinity has a representative E -valued 1-form that has a certain *standard form*. Namely there is a conformal automorphic

vector field on $\tilde{\Sigma}$ with canonical lift s_∞ such that $\omega = d\Pi^*s_\infty$ on the geometrically finite ends. On the rank two cusps, \mathcal{H}_0 , we can assume that ω is of the form given in Section 3.6. For the tubular neighborhoods of the cone singularity, \mathcal{C}_ϵ standard models for ω are given on p. 36 of [HK]. When ω is in standard form Theorem 3.5 implies that $\delta\omega$, $\text{tr Re } \omega$ and $\hat{d}(\text{tr Re } \omega)$ have finite L^2 -norm on the geometrically finite ends. The standard models on \mathcal{H}_0 and \mathcal{C}_ϵ are d of the canonical lift of a divergence free, harmonic vector field. In particular, $\delta\omega$ and $\text{tr } \omega$ are zero on \mathcal{H}_0 and \mathcal{C}_ϵ . Together this implies that $\delta\omega$, $\text{tr Re } \omega$ and $\hat{d}(\text{tr Re } \omega)$ have finite L^2 -norm on all of M .

We would like to show that every cohomology class in $H^1(M; E)$ is represented by a Hodge form. For example if $\tilde{\omega}$ is a closed E -valued and we could find a section τ of E such that

$$\Delta\tau = \delta\tilde{\omega} \tag{4.30}$$

then $\omega = \tilde{\omega} - d\tau$ we be closed and co-closed and in the same cohomology class as $\tilde{\omega}$.

To solve Equation (4.30) we view Δ as a linear operator on the Hilbert space of L^2 -sections of E . We can solve the equation if we can show that Δ is a self-adjoint operator with positive spectrum. Since Δ is an unbounded operator we need to restrict the domain of Δ .

To get a representative that is Hodge form it turns it out that we need to actually solve an equivalent equation in terms of real-valued 1-forms. Following [HK] we define

$$\text{dom } \hat{\Delta} = \{\alpha \in L^2 \mid \hat{d}\alpha, \hat{\delta}\alpha, \hat{d}\hat{\delta}\alpha, \hat{\delta}\hat{d}\alpha \in L^2\}$$

where all derivatives are defined as distributions. We then have the following theorem:

THEOREM 4.1. *On a hyperbolic cone-manifold $\hat{\Delta}$ is an elliptic, nonnegative, self-adjoint operator.*

Proof. As explained in the appendix of [HK] the result follows from the following Stokes' theorem. \square

THEOREM 4.2. *Let M be a hyperbolic cone-manifold. If smooth real valued forms α and β on M are in L^2 then*

$$(\hat{d}\alpha, \beta) = (\alpha, \hat{\delta}\beta). \tag{4.31}$$

Proof. If N is closed this is proved in [HK]. If \mathcal{C} is empty then the result is due to Gaffney [Ga]. More precisely Hodgson and Kerckhoff's work shows that if α and β have support on a compact neighborhood of the singular locus then (4.31) holds. Gaffney's work shows that if the support is the complement of a neighborhood of the singular locus than (4.31) holds. General α and β are the sums of forms of each type which implies Theorem 4.2 and hence Theorem 4.1. \square

We can now prove our Hodge theorem:

THEOREM 4.3. *Let M be a hyperbolic cone-manifold and $\tilde{\omega}$ a smooth E -valued 1-form in standard form representing a cohomology class in $H^1(M; E)$ that is conformal at infinity. Then there exists a unique Hodge form ω such that following holds:*

- (1) ω is cohomologous to $\tilde{\omega}$;
- (2) there exists an L^2 section s of E such that $ds = \tilde{\omega} - \omega$;
- (3) $\tilde{\omega} - \omega$ has finite L^2 -norm on $M \setminus \mathcal{C}_\epsilon$.

Proof. The proof is essentially the same as the proof of Theorem 2.7 in [HK]. We begin with a brief review of their proof and then fill in those details that their result does not provide.

Let \tilde{s} be an automorphic section such that $\tilde{\omega} = d\tilde{s}$. Any automorphic section can be written as the sum of a canonical lift and ι times an equivariant section. In particular there exists an automorphic vector field \tilde{v} and an equivariant vector field w such that $\tilde{s} = \tilde{V} - \iota \text{curl } \tilde{V} + \iota W$. Furthermore since $\tilde{\omega}$ is in standard form $w \equiv 0$ on $\mathcal{C}_\epsilon \cup \mathcal{H}_1$.

By Theorem 4.1 there is a unique vector field v_1 solving the equation

$$(\hat{\Delta} + 4)\tilde{v}_1 = \hat{\Delta}\tilde{v}$$

with \tilde{v}_1 in $\text{dom } \hat{\Delta}$. Since $\hat{\Delta}$ is elliptic, v_1 is smooth. If we let $v = \tilde{v} - v_1$ then by (2.7) v is harmonic so $\omega = d(V - \iota \text{curl } V)$ is co-closed. To finish the proof we need to show that ω is a traceless and hence a Hodge form and that ω satisfies (2) and (3).

By Theorem 2.2 the E -valued 1-form ω is traceless if $\text{div } v = 0$. Hodgson and Kerckhoff show that

$$(\hat{\Delta} + 4)\text{div } v = 0. \tag{4.32}$$

If we can show that $\text{div } v$ is in $\text{dom } \hat{\Delta}$ then we must have $\text{div } v = 0$ since by Theorem 4.1 $\hat{\Delta}$ has non-negative spectrum. By construction $\text{div } v = \text{div } \tilde{v} - \text{div } v_1$. Since $\tilde{\omega}$ is in standard form both $\text{div } \tilde{v}$ and $\hat{d}(\text{div } \tilde{v})$ are in L^2 . We also know that $\tilde{v}_1 \in \text{dom } \hat{\Delta}$ so $\text{div } v_1 = *\hat{d}\tilde{v}_1$ and $\hat{d}(\text{div } v_1)$ are in L^2 . Together this implies that $\text{div } v$ and $\hat{d}(\text{div } v)$ are in L^2 . By (4.32) $\hat{\Delta} \text{div } v = -4 \text{div } v$. Since $\text{div } v$ is in L^2 this implies that $\hat{\Delta} \text{div } v$ is also in L^2 . Therefore $\text{div } v$ is in $\text{dom } \Delta$ and must be zero.

We now prove (2). Let $s = V_1 - \iota \text{curl } V_1 - \iota W$. Then $ds = \tilde{\omega} - \omega$. We need to show that s is in L^2 on all of M . First we note that $\tilde{v}_1 \in \text{dom } \Delta$ so v_1 and $\widehat{\text{curl}} v_1 = -\frac{1}{2}*\hat{d}\tilde{v}_1$ are in L^2 on M . By Theorem 2.2, $w = \text{skew Re } \tilde{\omega}$. Since $\tilde{\omega}$ is in standard form, $\tilde{\omega}$ and therefore w are in L^2 on the geometrically finite ends. On $\mathcal{C}_\epsilon \cup \mathcal{H}_1$, $w \equiv 0$ so w is in L^2 on all of M proving (2).

To prove (3) we need the following lemma.

LEMMA 4.4. *If s is a section of E such that s and Δs are in L^2 on $M \setminus \mathcal{C}_\epsilon$, then ds is in L^2 on $M \setminus \mathcal{C}_\epsilon$.*

Proof. Let $f, g: M \rightarrow [0, 1]$ be smooth functions on M such that $f^2 + g^2 = 1$ and with $g = 1$ on $\mathcal{C}_{\epsilon/2}$ and $g = 0$ on $M \setminus \mathcal{C}_{\epsilon}$. Using standard techniques we can find smooth functions $f_n: M \rightarrow [0, 1]$ such that each f_n has compact support, $|df_n|$ is bounded and $f_n \rightarrow f$ uniformly on compact sets as $n \rightarrow \infty$. Recall that

$$(\alpha, \beta) = \int_M \alpha \wedge * \beta^\sharp.$$

We then have

$$\begin{aligned} (\Delta s, f_n^2 s) &= (ds, d(f_n^2 s)) \\ &= (ds, 2f_n s df_n) + (ds, f_n^2 ds) \\ &= (f_n ds, 2s df_n) + (f_n ds, f_n ds) \end{aligned}$$

where the first equality holds because $f_n^2 s$ has compact support. The inequality

$$\frac{1}{2}(f_n ds, f_n ds) + 2(s df_n, s df_n) \geq |(f_n ds, 2s df_n)|$$

gives us

$$\frac{1}{2}(f_n ds, f_n ds) \leq |(\Delta s, f_n^2 s)| + 2(df_n s, df_n s). \quad (4.33)$$

As $n \rightarrow \infty$, $(f_n ds, f_n ds) \rightarrow (f ds, f ds) \geq \|ds\|_{M \setminus \mathcal{C}_{\epsilon}}^2$ while the right hand side of (4.33) remains bounded since both $|f_n|$ and $|df_n|$ are bounded for all n . The lemma follows. \square

To finish the proof of (3) we note that by (2) s is in L^2 . Since $\tilde{\omega}$ is in standard form $\Delta s = \tilde{\omega} \tilde{\omega}$ is also in L^2 . Therefore Lemma 4.4 implies that ds is in L^2 on $M \setminus \mathcal{C}_{\epsilon}$ proving (3). \square

Remark. Lemma 4.4 is essentially due to Gaffney, [Ga]. The main difficulty is constructing the functions, f_n , through a distance function which may not be smooth. To make the functions smooth, Gaffney applies a smoothing operator to the distance function. The convex surfaces in the geometrically finite ends allow us to construct a smooth distance function directly.

A non-trivial simple closed curve γ on T_{ϵ} is a meridian if γ is homotopically trivial in N . An E -valued 1-form $\omega \in H^1(M; E)$ preserves the cone angle if the infinitesimal change in holonomy of γ induced by ω is trivial. The asymptotic behavior of ω is described in the following result of Hodgson and Kerckhoff [HK].

LEMMA 4.5. *Let M be a hyperbolic cone-manifold with all cone angles $\leq 2\pi$. If $\omega \in H^1(M; E)$ is an E -valued 1-form that preserves the cone angles, there exists $\epsilon_n \rightarrow 0$ such that*

$$\int_{T_{\epsilon_n}} \omega \wedge \omega^\sharp \rightarrow 0.$$

We now prove the main theorem of this section.

THEOREM 4.6. *Let M be a hyperbolic cone-manifold that is geometrically finite without rank one cusps and assume that all cone angles are $\leq 2\pi$. If $\omega \in H^1(M, E)$ is an E -valued 1-form that is conformal at infinity and preserves all cone angles and cusps then $\omega \sim 0$.*

Proof. By Theorem 4.3 we can assume that ω is a Hodge form and that ω is in L^2 on $M \setminus \mathcal{C}_\epsilon$. We will show that $\omega = 0$.

By Proposition 2.3

$$2 \int_{M_t} \|\omega\|^2 = \int_{\partial M_t} \iota\omega \wedge \omega^\sharp = B(t). \quad (4.34)$$

We will show that $B(t) \rightarrow 0$ as $t \rightarrow \infty$.

By Lemma 4.5 there exists $t_i \rightarrow \infty$ such that

$$\int_{T_{\epsilon/(1+t_i)}} \iota\omega \wedge \omega^\sharp \rightarrow 0$$

so we are left to analyze the boundary term on $S_t \cup H_t$.

Since ω is conformal at infinity and cusp preserving, Theorem 4.3 implies that ω is in L^2 on $M \setminus \mathcal{C}_\epsilon$ so

$$\begin{aligned} \int_{M \setminus \mathcal{C}_\epsilon} \omega \wedge *\omega^\sharp &= \int_{M_0} \omega \wedge *\omega^\sharp \\ &+ \int_0^\infty \int_{S_t \cup H_t} *(\omega \wedge *\omega^\sharp) dA_t dt \end{aligned}$$

is finite. (Note that $*(\omega \wedge *\omega^\sharp)$ is a smooth real function since $\omega \wedge *\omega^\sharp$ is a smooth real 3-form.) Therefore

$$\lim_{t \rightarrow \infty} \int_{S_t \cup H_t} *(\omega \wedge *\omega^\sharp) dA_t = 0.$$

We also have $|*_t(\iota\omega \wedge \omega^\sharp)| < *(\omega \wedge *\omega^\sharp)$ where $*_t$ is the Hodge $*$ -operator of the induced metric on $S_t \cup H_t$. Therefore

$$\left| \int_{S_t \cup H_t} \iota\omega \wedge \omega^\sharp \right| \leq \int_{S_t \cup H_t} |*_t(\iota\omega \wedge \omega^\sharp)| dA_t \leq \int_{S_t \cup H_t} *(\omega \wedge *\omega^\sharp) dA_t.$$

from which it follows $B(t) \rightarrow 0$ as $t \rightarrow \infty$. Taking the limit of (4.34) we see $2 \int_M \|\omega\|^2 = 0$ and therefore $\omega = 0$. \square

5. Representation Varieties of Cone-Manifolds

To understand local deformations of hyperbolic structures on a geometrically finite cone manifold we will study the representation variety of the fundamental groups of both the manifold and its boundary surfaces.

Let Γ be a finitely presented group and G a Lie group. Then $\mathcal{R}(\Gamma, G)$ is the space of representations of Γ in G . If Γ has n generators and m relations, r_i , then we can identify $\mathcal{R}(\Gamma, G)$ with a subset of G^n by

$$\mathcal{R}(\Gamma, G) = \{\gamma \in G^n : r_i(\gamma) = \text{id}, i = 1, 2, \dots, m\}.$$

If G is an algebraic group then $\mathcal{R}(\Gamma, G)$ is an algebraic variety.

We will be interested in the case where Γ is the fundamental group of a geometrically finite cone-manifold or a surface with a projective structure and $G = \text{PSL}_2\mathbb{C}$, the group of hyperbolic isometries and projective transformations. For simplicity of notation let $\mathcal{R}(M) = \mathcal{R}(\pi_1(M), \text{PSL}_2\mathbb{C})$ and $\mathcal{R}(S) = \mathcal{R}(\pi_1(S), \text{PSL}_2\mathbb{C})$ where M is a 3-manifold and S a closed surface.

The following theorem of Thurston, mentioned in the Introduction, is key to the existence of three-dimensional hyperbolic cone-manifolds.

THEOREM 5.1 ([TH], [CS]). *Let M be a compact hyperbolic manifold with boundary and holonomy representation ρ . Assume that the components of ∂M contain no spheres, t tori and surfaces of higher genus. If $T \subset \partial M$ is a torus, we also assume that $\rho(\pi_1(T)) \neq 1$. Then the dimension of the component of $\mathcal{R}(M)$ containing ρ is at least $t - 3\chi(M) + 3$.*

Hodgson and Kerckhoff proved the following result when ∂M contains only tori:

THEOREM 5.2 ([HK]). *Let M be a compact, connected 3-manifold with non-empty boundary consisting of t tori and surfaces of higher genus. Let $\rho \in \mathcal{R}(M)$ be an irreducible representation such that if T is a torus component of ∂M then $\rho(T) \neq 1$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If the natural map*

$$H^1(M, \partial M; E) \rightarrow H^1(M; E) \tag{5.35}$$

is zero, then at ρ , $\mathcal{R}(M)$ is a smooth complex manifold of dimension $t - 3\chi(M) + 3$.

Sketch of proof. To show that a variety is smooth one needs to show that the dimension of the Zariski tangent space is minimal. Theorem 5.1 gives a lower bound for this dimension so we need to show that the dimension at ρ equals this lower bound.

A fundamental result of Weil shows that $\dim TR(M)_\rho = H^1(M; E) + 3$ if ρ is irreducible. Hodgson and Kerckhoff show that if the natural map (5.35) is zero then

$$\dim H^1(M; E) = \frac{1}{2} \dim H^1(\partial M; E(\partial M)).$$

We are left to calculate $\dim H^1(\partial M; E)$ which will be the sum of the dimensions of $H^1(S; E)$ at ρ over all connected components S of ∂M . The dimension of $H^1(S; E)$

is well known. For a torus T with representation $\rho(T) \neq 1$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\dim H^1(T; E) = 2$. If S has genus > 1 at an irreducible representation, $\dim H^1(S; E) = -3\chi(S)$. Summing these dimensions we find

$$\dim H^1(M; E) = \frac{1}{2} \dim H^1(\partial M; E) = \frac{1}{2}(2t - 3\chi(\partial N)) = t - 3\chi(M).$$

Since the dimension of the tangent space at ρ is minimal, $\mathcal{R}(M)$ is smooth and has dimension $t - 3\chi(M)$. \square

Remark. To turn our sketch into an actual proof we need to view $\mathcal{R}(M)$ as a scheme instead of a variety. Then the Zariski tangent space of $\mathcal{R}(M)$ is the space of 1-cocycles with coefficients in the module $\text{Ad } \rho$. Furthermore at a representation ρ satisfying the conditions of Theorem 5.2 the algebraically defined Mumford quotient $\mathcal{R}(M)/\text{PSL}_2\mathbb{C}$ is isomorphic to the topological quotient $\mathcal{R}(M)/\text{PSL}_2\mathbb{C} = R(M)$. At the image of ρ , $R(M)$ will be a complex manifold whose (differentiable) tangent space is canonically identified with $H^1(M; E)$.

To apply this result to geometrically finite hyperbolic cone manifolds we need the following result:

PROPOSITION 5.3. *Let M be a hyperbolic cone-manifold that is geometrically finite without rank one cusps and let ρ be its holonomy representation.*

- (1) *The restriction of ρ to each geometrically finite end is irreducible.*
- (2) *ρ is irreducible.*
- (3) *Let T be the boundary of an ϵ -neighborhood of a component of the singular locus. Then the image of $\rho(\pi_1(T))$ is infinite and non-parabolic.*

Proof. 1. The holonomy of a projective structure on a surface of genus > 1 is always irreducible, for a reducible representation fixes a point on $\widehat{\mathbb{C}}$ and hence has image an affine group. Since a surface of genus > 1 cannot have an affine structure this is impossible. The restriction of ρ to a geometrically finite end is also the holonomy of a projective structure and therefore is irreducible.

2. If $\text{vol}(M)$ is finite then this is Lemma 4.6 in [HK]. If not M contains a geometrically finite end on which by (1) the holonomy is irreducible which implies that ρ is irreducible.

3. The holonomy of any homotopically non-trivial closed curve on T_ϵ that is not a multiple of the meridian will have hyperbolic holonomy. This implies that the image of $\rho(\pi_1(T))$ is infinite and non-parabolic.

The following corollary follows directly from Theorems 4.6 and 5.2 along with Proposition 5.3.

COROLLARY 5.4. *Let ρ be the holonomy representation of a hyperbolic cone-manifold M that is geometrically finite without rank one cusps. If all cone angles of M*

are $\leq 2\pi$ then $R(M)$ is smooth at ρ with dimension $n + m - 3\chi(M)$ where n is the number of components of \mathcal{C} and m is the number of rank two cusps.

For $\gamma \in \pi_1(M)$ let $\mathcal{L}_\gamma(\rho)$ denote the complex length of $\rho(\gamma)$. If $\rho(\gamma)$ is hyperbolic then $\mathcal{L}_\gamma(\rho)$ is the sum of the translation length plus i times the angle of rotation. While this is only well defined up to sign and the angle is only defined modulo 2π after making an initial choice \mathcal{L}_γ extends to a holomorphic function in a neighborhood of ρ . In our setting when γ is the meridian of a cone singularity it is natural to choose $\mathcal{L}_\gamma(\rho)$ to be the cone angle. If $\rho(\gamma)$ is parabolic we define $\mathcal{L}_\gamma(\rho) = 0$. In this case there is no way to make a choice of sign. Instead we view \mathcal{L}_γ as a map to $\mathbb{C}/\{\pm 1\}$. Although this will allow us to extend \mathcal{L}_γ to a continuous map in neighborhood of ρ it will not in general be differentiable. For this reason at parabolic elements it is convenient to use the trace map. That is $\text{Tr}_\gamma(\rho)$ assigns to each $\rho \in R(M)$ the trace of $\rho(\gamma)$. Again this map is only defined up to sign but at a parabolic the trace is ± 2 so a well defined choice of sign can be made. The trace then extends to a holomorphic map at parabolics. Note that $\text{Tr}_\gamma(\rho) = 2 \cos h(\mathcal{L}_\gamma(\rho)/2)$.

To understand the derivative of \mathcal{L}_γ (or Tr_γ) it is helpful to look at the bundle $E(\gamma)$ which we define to be the restriction of E to a smooth loop in the free homotopy class of γ . Then each cohomology class $\omega \in H^1(M; E)$ restricts to a cohomology class in $H^1(\gamma; E(\gamma))$. If $\rho(\gamma)$ is not the identity then $H^1(\gamma; E(\gamma)) \cong \mathbb{C}$ where the natural isomorphism sends cohomology classes to tangent vectors to the space of complex lengths (or traces). Note that the infinitesimal change in holonomy of γ induced by ω is trivial if and only if ω restricts to a trivial element of $H^1(\gamma; E(\gamma))$. More precisely we have the following lemma which is essentially contained in Theorem 4.5 in [HK].

LEMMA 5.5. (1) *Let γ be a meridian of the cone singularity. Then $(\mathcal{L}_\gamma)_* \omega = 0$ if and only if ω preserves the cone angle.*

(2) *Let γ be homotopic to a rank two cusp. Then $(\text{Tr}_\gamma)_* \omega = 0$ if and only if ω is cusp preserving.*

Note that if the cone angle 2π the holonomy of the meridian will be the identity. This special case is also dealt with in Theorem 4.5 of [HK].

We now describe a local parameterization of $R(M)$ that is the main theorem of this paper. To do so we need to recall some basic facts about the space of marked projective structures, $P(S)$, on a closed surface S of genus > 1 . These can all be found in [Gun]. $P(S)$ is a complex manifold of dimension $-3\chi(S)$. If $\Sigma \in P(S)$ is a projective structure then the tangent space of $P(S)$ at Σ can be canonically identified with $H^1(\Sigma; E(\Sigma))$. The *Teichmüller space*, $T(S)$, of S is the space of marked conformal structures on S . Since a projective structure also defines a conformal structure there is a projection, $p: P(S) \rightarrow T(S)$. Furthermore, if $\omega \in H^1(\Sigma; E(\Sigma))$ is an $E(\Sigma)$ -valued 1-form then $p_* \omega = 0$ if and only if ω is conformal. There is also a holonomy map, $h: P(S) \rightarrow R(S)$. We will need the following theorem:

THEOREM 5.6 (Hejhal[Hej]). *The map h is a holomorphic, local homeomorphism.*

Assume M is a hyperbolic cone-manifold that is geometrically finite without rank one cusps. Assume the cone singularity has n components and that M has m rank two cusps. Let S be the union of the higher genus boundary components of ∂M . $R(S)$, $P(S)$ and $T(S)$ will be the product of the representation varieties, spaces of projective structures and Teichmüller spaces, respectively, of the components of S . For each component of the cone singularity we let \mathcal{L}_i , $i = 1, \dots, n$, be the complex length of the meridian. For each rank two cusp we choose a generator of the corresponding $\mathbb{Z} \oplus \mathbb{Z}$ subgroup and let \mathcal{L}_i be its complex length and Tr_i its trace with $i = n + 1, \dots, n + m$. We also have a map, $\partial: R(M) \rightarrow R(S)$, that restricts each representation to a representation of the boundary surfaces. We then define a maps, Φ and $\bar{\Phi}$ by

$$\Phi(\sigma) = (\mathcal{L}_1(\sigma), \dots, \mathcal{L}_n(\sigma), \text{Tr}_{n+1}(\sigma), \dots, \text{Tr}_{n+m}(\sigma), p \circ h^{-1} \circ \partial(\sigma))$$

and

$$\bar{\Phi}(\sigma) = (\mathcal{L}_1(\sigma), \dots, \mathcal{L}_{n+m}(\sigma), p \circ h^{-1} \partial(\sigma))$$

for $\sigma \in R(M)$. Note that the image of Φ is contained in $\mathbb{C}^{n+m} \times T(S)$ while the image of $\bar{\Phi}$ is contained in $\mathbb{C}^n \times (\mathbb{C}/\pm 1)^m \times T(S)$.

We now prove our main theorem.

THEOREM 5.7. *Assume M is a geometrically finite hyperbolic cone-manifold without rank one cusps and with holonomy representation ρ . If all cone angles of M are $\leq 2\pi$ then Φ is a holomorphic, local homeomorphism and $\bar{\Phi}$ is a local homeomorphism at ρ .*

Proof. By Corollary 5.4, $R(M)$ is a smooth complex manifold of dimension $n + m - 3\chi(M)$ which is equal to the dimension of $\mathbb{C}^{n+m} \times T(S)$. Since \mathcal{L}_i , Tr_i and $p \circ h^{-1}$ are all holomorphic Φ is holomorphic. To show that Φ is a local homeomorphism we need to show that Φ_* has trivial kernel. If $\omega \in H^1(M; E)$ is an E -valued 1-form such that $\Phi_*\omega = 0$ then $(\mathcal{L}_i)_*\omega = 0$, $(\text{Tr}_i)_*\omega = 0$ and $p_*\omega = 0$. By Lemma 5.5, $(\mathcal{L}_i)_*\omega = 0$ implies that ω preserves the cone angle and $(\text{Tr}_i)_*\omega = 0$ implies that ω is cusp preserving. Finally if $p_*\omega = 0$, then ω is conformal at infinity. Therefore Theorem 4.6 implies that ω is trivial so Φ_* has trivial kernel and Φ is a local homeomorphism at ρ . The relationship between the trace and the complex length then implies that $\bar{\Phi}$ is a local homeomorphism. \square

This parameterization leads to our local rigidity theorem.

THEOREM 5.8. *If M is a geometrically finite cone-manifold without rank one cusps and all cone angles are $\leq 2\pi$ then M is locally rigid rel cone angles and the conformal boundary.*

Proof. Let M_t be a smooth family of cone-metrics on M such that $M_0 = M$ and such that the conformal structures at infinity and cone angles of M_t agree with those of M . Then by Theorem 5.7 the holonomy representations ρ_t for M_t are equal to ρ_0 .

Theorem 1.7.1 of [CEG] implies that for every compact core, M' , of M_0 there exists a t' such that M' isometrically embeds in M_t for $t < t'$. Choose M' such that $\partial M'$ is a collection of convex surfaces of higher genus and Euclidean tori around each rank two cusp and component of the singular locus. Then any isometry of M' into M_t can be extended to an isometry from M_0 onto M_t . Hence M_0 is locally rigid rel cone angles and the conformal boundary. \square

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