Homework 2, Math 5510
September 22, 2015
Section 18: 3, 7(a)
Section 19: 2, 7, 8, 10
Section 20: 3, 4, 8

18.3(a) If $U \in \mathcal{T}$ and i is continuous then $i^{-1}(U) = U$ is in $\mathcal{T}'$. Hence $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T}$ is finer than $\mathcal{T}'$. On the other hand if $\mathcal{T}'$ is finer then $\mathcal{T}$ then if $U \in \mathcal{T}$ then $U \in \mathcal{T}'$ so $i^{-1}(U)$ is open in $\mathcal{T}'$ if it is open in $\mathcal{T}$ and therefore i is continuous.

18.3(b) By (a) i is continuous if and only if $\mathcal{T}'$ is finer than $\mathcal{T}$. Similarly i^{-1} is continuous if and only if $\mathcal{T}$ is finer than $\mathcal{T}'$. It follows that i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.

18.7(a) Let (a, b) be an interval in $\mathbb{R}$. As these sets form a basis for $\mathbb{R}$ we just need to show that the $f^{-1}(a, b)$ is open in $\mathbb{R}_\ell$. For $x \in f^{-1}(a, b)$ we claim there exists an $\epsilon > 0$ such that if $y \in [x, x + \epsilon)$ then $f(y) \in (a, b)$. Assume not. Then for each n there exists a $x_n \in [x, x + 1/n)$ such that $f(x_n) \notin (a, b)$. But x_n limits to x from the right so $f(x_n) \to f(x)$ and since $f(x) \in (a, b)$ and (a, b) is open we must have $f(x_n) \in (a, b)$ for large n, a contradiction. Therefore there exists an $\epsilon > 0$ such that $[x, x + \epsilon) \subset f^{-1}(a, b)$. As half open intervals are a basis for $\mathbb{R}_\ell$ this shows that $f^{-1}(a, b)$ is open in $\mathbb{R}_\ell$ and f is continuous.

19.3 We need to show that the box topology on $\prod A_\alpha$ is equivalent to the subspace topology. Let $U \subset \prod A_\alpha$ be open in the box topology. Then for each $x \in U$ there exists a basis element $B = \prod B_\alpha$ (in the box topology) with $x \in B \subset U$. The B_α are open subsets in the subspace topology for each A_α so $B_\alpha = V_\alpha \cap A_\alpha$ for an open set V_α in X_α. Then $B = (\prod V_\alpha) \cap (\prod A_\alpha)$ so B is open in the subspace topology for $\prod A_\alpha$ and since for every $x \in U$ there is open set B in the subspace topology such that $x \in B$ and $B \subset U$ we have that U is open in the subspace topology.

Now assume that U is open in the subspace topology. Then $U = V \cap (\prod A_\alpha)$ for an open subset V in $\prod X_\alpha$. Then for all $x \in V$ there exists a basis element $B = \prod B_\alpha$ for the box topology with $x \in B \subset V$. Then $B \cap \prod A_\alpha = \prod (B_\alpha \cap A_\alpha)$ is open in the box topology for $\prod A_\alpha$ so for $x \in U$ have found an open set B in the box topology such that $x \in B \subset U$ and therefore U is open in the box topology.

The proof for the product topology is almost exactly the same if we simply replace the word "box" with "product" in the above paragraphs with only one subtly in each paragraph. For the first paragraph we note that for all but finitely many α we have that $B_\alpha = A_\alpha$ and in all these cases we can choose $V_\alpha = X_\alpha$ so that $\prod V_\alpha$ is a basis element of the product topology. For the second paragraph we see that for all but finitely many of the B_α we have $B_\alpha = X_\alpha$ so $B_\alpha \cap A_\alpha = A_\alpha$ so the product $\prod(B_\alpha \cap A_\alpha)$ is a basis element, and open, in the product topology on $\prod A_\alpha$.

19.7 In the box topology $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$. To see this assume that $x \in \mathbb{R}^\omega \backslash \mathbb{R}^\infty$. Then $x = (x_i)$ will have infinitely many non-zero terms which we index x_{i_j}. Let $\epsilon_{i_j} = |x_{i_j}|/2$. For i not in the subsequence $\{i_j\}$ we let $\epsilon_i = 1$ (although any positive number will do) and set $U = \prod(x_i - \epsilon_i, x_i + \epsilon_i)$. Then U is an open neighborhood of x in the box topology. If $y \in U$ then $y_{i_j} \neq 0$ and since $\{i_j\}$ is an infinite sequence this implies that $U \cap \mathbb{R}^\infty = \emptyset$. Therefore by Theorem 17.5 (a), x is not in the closure of $\mathbb{R}^\infty$.

For the product topology the closure of $\mathbb{R}^\infty$ is all of $\mathbb{R}^\omega$. Let U be a neighborhood in the product topology of a point $x \in \mathbb{R}^\omega$. Then there is a basis element $B = \prod U_i$. But for all but finitely many i, $U_i = \mathbb{R}$ and therefore $0 \in U_i$. Therefore B intersects $\mathbb{R}^\infty$ and by Theorem 17.5 (a), $x \in \overline{\mathbb{R}^\infty}$. Since x was arbitrary we have that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

19.8 Let $f_i : \mathbb{R} \to \mathbb{R}$ be a function indexed by $i \in \mathbb{N}$. Then if $f : \mathbb{R}^\omega \to \mathbb{R}^\omega$ is a function with $f(x) = (f_1(x_1), f_2(x_2), \dots)$ and $A = \prod A_i$ is a product subset of $\mathbb{R}^\omega$ then $f^{-1}(A) = \prod f_i^{-1}(A_i)$. In particular if each f_i is continuous then so is f in both the box and product topologies since if A is a basis element then so is $f^{-1}(A)$.

For the function h we let $h_i : \mathbb{R} \to \mathbb{R}$ be defined by $h_i(x) = a_i x + b_i$. Then h is of the form above and is continuous. Furthermore the functions h and h_i are invertible with $h^{-1}(x) = (h_1^{-1}(x_1), h_2^{-1}(x_2), \dots)$ so h^{-1} is also continuous in both topologies and therefore h is a homeomorphism in both topologies.

19.10 (a) Let $\{\mathcal{T}_\beta\}$ be the collection of topologies on A such that all the f_α are continuous. This set of topologies is non-empty since it contains the discrete topology on A. Let $\mathcal{T} = \cap \mathcal{T}_\beta$. By Problem ... from the last homework that $\mathcal{T}$ is a topology. We claim that this the coarsest topology where all the f_α are continuous. To see that all the f_α are continuous we let U be an open set in X_α. Then $f_\alpha^{-1}(U)$ is open in all the $\mathcal{T}_\beta$ so is contained in $\mathcal{T}$ and hence f_α is continuous in the $\mathcal{T}$ topology. The topology $\mathcal{T}$ must be the coarsest where all the f_α are continuous as it is contained in every topology where the f_α are continuous.

19.10 (b) Clearly $\mathcal{S} \subset \mathcal{T}$ as for all the f_α to be continuous the sets in $\mathcal{S}$ must be open. We saw in class that the topology generated by $\mathcal{S}$ is the coarsest topology that contains $\mathcal{S}$ so by (a) this must be $\mathcal{T}$.

19.10 (c) The composition of continuous function is continuous so if $g : Y \to A$ is continuous then $f_\alpha \circ g$ is continous for all $\alpha \in J$. For the other direction let $U \in \mathcal{T}$ be an open subset of A and let $y \in g^{-1}(U)$. Since $\mathcal{S}$ is a subbasis there exists sets $U_1, \dots, U_n$ with each U_i open in some X_{α_i} and $f(x) \in f_{\alpha_1}^{-1}(U_1) \cap \dots \cap f_{\alpha_n}^{-1}(U_n) \subset U$. Since all $f_\alpha \circ g$

are continuous we have that $(f_{\alpha_1} \circ g)^{-1}(U_1) \cap \cdots \cap (f_{\alpha_n} \circ g)^{-1}(U_n)$ is open. As it contains x and is contained in $g^{-1}(U)$ this implies that $g^{-1}(U)$ is open.

19.10 (d) Let $U \in \mathcal{T}$ be an open set in A and let $x \in U$. We will show that there exists an open set V in $\prod X_\alpha$ such that $V \cap Z \subset f(U)$ and $f(x) \in V \cap Z$. By (b), $\mathcal{S}$ is a subbasis for $\mathcal{T}$ so we can find $\alpha_1, \dots, \alpha_n \in J$ and open sets U_{α_i} in X_{α_i} such that the basis element $B = f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap f_{\alpha_n}^{-1}(U_{\alpha_n})$ contains x and is contained in U. Now let $U_\alpha = X_\alpha$ if $\alpha \notin \{\alpha_1, \dots, \alpha_n\}$. Then $V = \prod U_\alpha$ is open in $\prod X_\alpha$ and $Z \cap V = f(B)$ is open in Z and contains x. Hence $f(U)$ is open.

20.3 (a) Let $V \subset \mathbb{R}$ be open and $(x, y) \in d^{-1}(V)$. Then there exists an $\epsilon > 0$ such that $(d(x, y) - \epsilon, d(x, y) + \epsilon) \subset V$. Then $U = (x - \epsilon/2, x + \epsilon/2) \times (y - \epsilon/2, y + \epsilon/2)$ is a neighborhood of (x, y) and $d(U) \subset V$ so by Theorem 18.1(d), d is continuous.

20.3 (b) We'll show that an arbitrary basis element for the metric topology is open in $X' \times X'$. Let $(x, y) \in X' \times X'$ and $\epsilon_x, \epsilon_y > 0$. Then $B = B_d(x, \epsilon_x) \times B_d(y, \epsilon_y)$ is a basis element for the metric topology on $X \times X$. Let $(x', y') \in B$ and $t = d(x', y')$. Let $\epsilon' = \min\{d(x, x'), d(y, y')\}$. Then $d^{-1}((t - \epsilon', t + \epsilon'))$ is open in $X' \times X'$ so B is open in $X' \times X'$.

20.4 (a) Since each coordinate function is continuous, all three functions are continuous in the product topology. The set $U = \prod(-1, 1)$ is open in the uniform topology but $f^{-1}(U) = \{0\}$ is not open so f is not continuous in the uniform topology. To show that g is continuous we observe that for any basis element $B_{\bar{\rho}}(y, \epsilon)$ in the uniform topology and any $x \in g^{-1}(B_{\bar{\rho}}(y, \epsilon))$ there is a ball $B_d(x, \delta) \subset g^{-1}(B_{\bar{\rho}}(y, \epsilon))$ where $\delta < \epsilon - \bar{\rho}(y, f(x))$. Therefore $g^{-1}(B_{\bar{\rho}}(y, \epsilon))$ is open and g is continuous. In fact for any function $r(t) = (\lambda_1 t, \lambda_2 t, \dots)$ with the $|\lambda_i| \leq 1$ we have that $B_d(x, \delta) \subset r^{-1}(B_{\bar{\rho}}(y, \epsilon))$ if $\delta < \epsilon - \bar{\rho}(y, f(x))$ so h is also continuous.

In the box topology know of the functions are continuous. To see this we observe that the pre-image of $\prod(-1/n^2, 1/n^2)$ is an open set whose pre-image under all three functions is $\{0\}$ which is not open.

20.4 (b) We first make some general comments. In all three topologies a necessary condition for a sequence to converge is that it must converge in each coordinate. If we show this for the product topology than it is automatically true in the uniform and box topologies as they are finer topologies. Let $a_i = (a_i^1, a_i^2, \dots)$ be a sequence in $\mathbb{R}^\omega$ that converges to $a \in \mathbb{R}^\omega$. If $\bar{D}(a_i, a) < \epsilon$ then for all $j \in \mathbb{N}$, $|a_i^j - a^j|/j < \epsilon$ and this implies that $a_i^j \to a^j$ as $i \to \infty$, as desired. Therefore if any of the 4 sequences converge, they converge to $0 = (0, 0, \dots)$.

Next we notice that this is also a sufficient condition for convergence in the product topology. Let a_i be as above and assume that $a_i^j \to a^j$ as $i \to \infty$. Let $U = \prod U_j$ be a basis element for the product topology. We can assume that for $j > J$ that $U_j = \mathbb{R}$. For each $j \leq J$, for all but finitely many i, $a_i^j \in U_j$ and therefore for all but finitely many

i, $a_i^j \notin U_j$ for all $1 \leq j \leq J$. This implies that for all but finitely many i, $a_i \in U$ and $a_i \to a$. This implies that all 4 sequences converge in the product topology.

In the uniform topology we notice that $\bar{\rho}(w_i, 0) = i$ so w_i doesn't converge. However, for the other three topologies we have $d(x_i, 0) = d(y_i, 0) = d(z_i, 0) = 1/i$ so these three sequences convergence.

For the box topology a sequence a_i is convergent if and only if it converges in each coordinate and there exists an N and a J such that if $i > N$ then sequence a_i^j is constant once $j > J$. For the necessity of this condition we prove the contrapositive and assume that sequences converges in each coordinate to a^j but that the second condition doesn't hold. Therefore we can find an increasing subsequences of integers i_n and j_n such that $a_{i_m}^{j_m} \neq a^{j_m}$. We then choose an open neighborhood $U = \prod U_i$ of $a = (a^1, a^2, \dots)$ with $U_{j_m} = (a_m^j - \epsilon_m, a_m^j + \epsilon_m)$ where $\epsilon_m = |a_{i_m}^{j_m} - a^{j^m}|/2$. Then none of the a_{i_m} will be contained in U so $a_i \notin a$. This implies that the sequence w_i, x_i and y_i do not converge in the box topology.

The sufficiency of this condition is a little tedious so we will just show that z_i converges. Let $\prod U_i$ be a basis element of the box topology that contains $(0, 0, \dots)$. Then $U_1 \cap U_2$ is open and contains 0 so there exists an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subset U_1 \cap U_2$. Therefore if $1/i < \epsilon$, we have $z_i \in \prod U_i$ so z_i converges to $(0, 0, \dots)$.

20.8 (a) We first show that the box topology is finer than the ℓ^2-topology. Let $B_{\ell^2}(x, \epsilon)$ be an ℓ^2-ball and $y \in B_{\ell^2}(x, \epsilon)$. Then there exists an $\epsilon' > 0$ such that $B_{\ell^2}(y, \epsilon') \subset B_{\ell^2}(x, \epsilon)$. Choose $\delta_i > 0$ such that $\sum \delta_i^2 < (\epsilon')^2$. Then

$$\prod (y_i + \delta_i, y_i - \delta_i) \subset B_{\ell^2}(y, \epsilon') \subset B_{\ell^2}(x, \epsilon)$$

so the box topology is finer than the ℓ^2-topology.

Now we show the ℓ^2-topology is finer than the uniform topology. Let $y \in B_{\bar{\rho}}(x, \epsilon)$ and $\epsilon' > 0$ such that $B_{\bar{\rho}}(y, \epsilon') \subset B_{\bar{\rho}}(x, \epsilon)$. But $B_{\ell^2}(y, \epsilon') \subset B_{\bar{\rho}}(y, \epsilon') \subset B_{\bar{\rho}}(x, \epsilon)$ so the uniform topology is finer than the ℓ^2-topology.

20.8 (b) Let a_i be the sequence in $\mathbb{R}^\infty$ where the ith coordinate is i and all other coordinates are 0. By the convergence criteria from Problem 20.4(b) we see that a_i converges to $(0, 0, \dots)$ in the product topology. In the uniform topology $\bar{\rho}(0, a_i) - i$ so a_i doesn't converge and the uniform and product topologies are distinct.

Let b_i be the sequence in $\mathbb{R}^\infty$ where the first i coordinates are $1/\sqrt{i}$. Then $\bar{\rho}(0, b_i) = 1/\sqrt{i}$ but $d_{\ell^2}(0, a_i) = 1$. So a_i converges in the uniform topology but not in the ℓ^2-topology so the two topologies are distinct.

Let c_i be the sequence in $\mathbb{R}^\infty$ where the ith coordinate is $1/i$. Then by our divergence criteria from Problem 20.4(b), c_i diverges in the box topology. But $d_{\ell^2}(0, c_i) = 1/\sqrt{i}$ so c_i converges to $(0, 0, \dots)$ in the ℓ^2 topology. Hence these two topologies are also distinct.

20.8 (c) The sequence c_i is in the Hilbert cube H. As it converges in the ℓ^2 topology but not in the box topology these two topologies are different on H.

Let $y \in B_{\ell^2}(x,\epsilon) \cap H$ and as above choose $\epsilon' > 0$ such that $B_{\ell^2}(y,\epsilon') \cap H \subset B_{\ell^2}(x,\epsilon) \cap H$. Choose $N > 0$ such that $\sum_{n>N} 1/n^2 \leq (\epsilon')^2/4$ and let $\delta = \epsilon'/\sqrt{2N}$. Let $U_i = (y_i - \delta, y_i + \delta)$ for $i = 1, \ldots, N$ and $U_i = \mathbb{R}$ for $i > N$. Then $(\prod U_i) \cap H \subset B_{\ell^2}(y,\epsilon') \cap H \subset B_{\ell^2}(x,\epsilon) \cap H$ so the product topology on H is finer than ℓ^2-topology and hence the two topologies are equal. Since the uniform topology is between the ℓ^2 and product topology all three topologies are equal.