Name:

Midterm 3, Math 3210
November 20th, 2015

You must write in complete sentences and justify all of your work. Do 3 of the 4 problems below. All 3 problems that you do will be equally weighted.

1. Let $f\colon (a,b) \to \mathbb{R}$ be uniformly continuous and assume that $g\colon (a,b) \to \mathbb{R}$ is another function such that there exists a $K > 0$ with

   $$|g(x) - g(y)| \le K|f(x) - f(y)|$$

   for all $x, y \in (a,b)$. Show that g is uniformly continuous.

   Solution: Fix $\epsilon > 0$. Since f is uniformly continuous there exists a $\delta > 0$ such that if $x, y \in (a,b)$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon/K$. Then

   $$|g(x) - g(y)| \le K|f(x) - f(y)| < K(\epsilon/K) = \epsilon$$

   so g is uniformly continuous.

2. Define functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ by

   $$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$$

   and

   $$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$$

   For both functions either find the derivative at 0 (with proof) or show that it doesn't exist.

   Solution: The derivative $f'(0)$ exists if the limit

   $$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

   exists. To check this we evaluate the left and right-handed limits and see if they are equal. First the left-handed limit it

   $$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{0}{x} = 0.$$

The right-handed limit is

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x}{x} = 1.$$

As the right and left-handed limits aren't the same the derivative doesn't exist.

We similarly check the left and right hand limits of

$$\lim_{x \to 0} \frac{g(x) - f(0)}{x - 0}$$

exists. The left-handed limit is

$$\lim_{x \to 0^-} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{0}{x} = 0.$$

The right-handed limit is

$$\lim_{x \to 0^-} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0.$$

As the right and left-handed limits are both 0 we have $g'(0) = 0$.

3. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function that is differentiable on $(0,1)$. Assume that $f(0) = 1$ and $f'(x) > -1$ for all $x \in (0,1)$. Show that $f(x) > 0$ for all $x \in [0,1]$.

   Solution: For all $x \in (0,1]$ there exists a $c \in (0,x)$ such that

   $$\frac{f(x) - f(0)}{x - 0} = f'(c)$$

   and therefore

   $$\frac{f(x) - 1}{x} > -1.$$

   Rearranging we have $f(x) > -1(x) + 1 \geq 0$ since $x \leq 1$.

4. Define $f\colon [0,1] \to \mathbb{R}$ by

   $$f(x) = \begin{cases} 0 & \text{if } x < 1/2 \\ 1 & \text{if } x \geq 1/2 \end{cases}$$

   and define partions $P_n = \{0 < \frac{1}{2} - \frac{1}{n} < \frac{1}{2} + \frac{1}{n} < 1\}$. Calculate $U(f, P_n)$ and $L(f, P_n)$. Is f integrable? Make sure to justify your answer.

   Solution: The upper sum is

   $$U(f, P_n) = 0((1/2 - 1/n) - 0) + 1(2/n) + 1(1 - (1/2 + 1/n)) = 1(1/2 + 1/n)$$

   and the lower sum is

   $$L(f, P_n) = 0((1/2 - 1/n) - 0) + 0(2/n) + 1(1 - (1/2 + 1/n)) = 1(1/2 - 1/n).$$

   As $n \to \infty$ we have $U(f, P_n) - L(f, P_n) = 2/n \to 0$ so f is integrable.