Name:

Midterm 2, Math 3210
October 23rd, 2015

You must write in complete sentences and justify all of your work. Do 3 of the 4 problems below. All 3 problems that you do will be equally weighted. Clearly mark in the table below which 3 problems you want graded.

1. Directly using the definition of a limit show that $\lim_{n\to\infty} \sqrt{n^2+1} - n = 0$.

   Solution: Fix $\epsilon > 0$ and choose $N > \frac{1}{2\epsilon}$. Then if $n > N$ we have

$$\begin{aligned} \left|(\sqrt{n^2+1} - n) - 0\right| &= \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n} \\ &= \frac{1}{\sqrt{n^2+1} + n} \\ &\le \frac{1}{\sqrt{n} + n} \\ &= \frac{1}{2n} < \frac{1}{2N} < \epsilon \end{aligned}$$

   so $\sqrt{n^2+1} - n \to 0$

2. Let a_n be a sequence of positive numbers and assume that the sequence $b_n = a_n/n$ converges to some $b > 0$. Show that there exists a constant $c > 0$ such that $a_n \ge cn$ for all positive integers n.

   Solution: In Problem 2.3.8 from Homework 5 we saw that there exists a $c > 0$ such $b_n > c$ for all n then $a_n/n = b_n > c$ so $a_n > cn$ for all n.

   Here is another approach. Assume the statement is false. Then for all $c > 0$ there exists an n such that $a_n < cn$. In particular, for each positive integer k there is an n_k such that $a_{n_k} < (1/k)n_k$. Now look at the subsequence $b_{n_k} = a_{n_k}/n_k < 1/k$. Then $b_{n_k} \to 0$. This is a contradiction since as b_n converges to a positive number so must every subsequence of b_n.

3. Directly using the definition of a Cauchy sequence show that $a_n = \frac{1}{2n}$ is a Cauchy sequence.

Solution: Fix $\epsilon > 0$ and choose $N > \frac{1}{2\epsilon}$. Then if $n \geq m > N$ we have

$$\begin{aligned} |a_n - a_m| &= \left| \frac{1}{2n} - \frac{1}{2m} \right| \\ &\leq \frac{1}{2n} \\ &< \frac{1}{2N} < \epsilon. \end{aligned}$$

Therefore a_n is a Cauchy sequence.

4. Let $f : [0,1] \to [-1,0]$ be a continuous function. Show that there exists an $x \in [0,1]$ such that $f(x) = -x$.

   Solution: Let $g(x) = f(x) + x$. If $f(0) = 0$ or $f(1) = -1$ we are done so we can assume that $f(0) < 0$ and $f(1) > -1$ (since $f(x) \in [-1,0]$). Then $g(0) = f(0) < 0$ and $g(0) = f(1) + 1 > -1 + 1 = 0$ so by the Intermediate Value Theorem there exists and $x \in (0,1)$ such that $g(x) = 0$. But then $g(x) = f(x) + x = 0$ and $f(x) = -x$ as desired.