Midterm 1, Math 3210
September 16, 2015
You must write in complete sentences and justify all of your work.

1. (10 pts.) Use induction to prove that $8^n - 5^n$ is divisible by 3 for all $n \in \mathbb{N}$.

   Solution: When $n = 1$ we have $8^1 - 5^1 = 8 - 5 = 3$ which is divisible by 3. Now we assume that $8^n - 5^n$ is divisible by 3 and prove that $8^{n+1} - 5^{n+1}$ is divisible by 3. By adding and subtracting $8 \cdot 5^n$ to the expression we see that

   $$\begin{aligned} 8^{n+1} - 5^{n+1} &= 8^{n+1} - 8 \cdot 5^n + 8 \cdot 5^n - 5^{n+1} \\ &= 8(8^n - 5^n) + 5^n(8-5) \\ &= 8(8^n - 5^n) + 3 \cdot 5^n. \end{aligned}$$

   The last expression is divisible by 3 since the first term contains $8^n - 5^n$ and is divisible by 3 by our induction assumption and the second term is a product with 3. Therefore $8^{n+1} - 5^{n+1}$ is divisible by 3.

   By induction we have shown that $8^n - 5^n$ is divisible by 3 for all $n \in \mathbb{N}$.

2. Let F be a field as defined in the book (and in the notes). Given $x, y, z \in F$ show that:

   (a) (10 pts.) If $x + z = y + z$ then $x = y$.

   (b) (5 pts.) $x \cdot 0 = 0$.

   In your proofs you can only use the properties of a field given in the notes. Make sure you clearly indicate which field properties you are using as you use them.

   Solution: See Example 1.3.2 in the book.

3. (10 pts.) Let $L = \{r \in \mathbb{Q} | r^3 < 2\}$. Show that L is a Dedekind cut.

   Solution: We first note at since $1^3 = 1 < 2$, we have that $1 \in L$ so $L \neq \emptyset$. Since $2^3 = 8 > 2$, $2 \notin L$ and $L \neq \mathbb{Q}$.

   Next we show that L has no largest element. Assume that $r \in L$. Since $r^3 < 2$ we have that $r < \sqrt[3]{2}$. By the Archimedean Principle there exists an $r' \in \mathbb{Q}$ with $r < r' < \sqrt[3]{2}$. Since $(r')^3 < 2$, $r' \in L$ and L has no largest element.

   Finally we show that if $r \in L$ and $r' \in \mathbb{Q}$ with $r' < r$ then $r' \in L$. Since $r' < r$, we have that $(r')^3 < r^3$. Since $r \in L$ we have that $r < 2$. Together this implies that $(r')^3 < r^3 < 2$ so $r' \in L$.