**Notes for Math 3210, Final**

***Limits.*** Let $\{a_n\}$ be a sequence. Then

$$\lim a_n = a$$

if for all $\epsilon > 0$ there exists an $N$ such that if $n > N$ then $|a_n - a| < \epsilon$. If no such $a$ exists then the sequence is *divergent*. The sequence $a_n$ is *Cauchy* if for all $\epsilon > 0$ there exists an $N > 0$ such that if $n, m > N$ then $|a_n - a_m| \leq \epsilon$.

**Theorem 0.1** *A sequence is convergent if and only if it is Cauchy.*

**Theorem 0.2** *Every bounded sequence of real numbers has a convergent subsequence.*

**Theorem 0.3** *Suppose $a_n \to a$, $b_n \to b$, $c$ is a real number and $k$ a natural number. Then*

1. $ca_n \to ca$;
2. $a_n + b_n \to a + b$;
3. $a_n b_n \to ab$;
4. $a_n/b_n \to a/b$ *if* $b \neq 0$ *and* $b_n \neq 0$ *for all* $n$;
5. $a_n^k \to a^k$;
6. $a_n^{1/k} \to a^{1/k}$ *if* $a_n \geq 0$ *for all* $n$.

If $A$ is a subset of $\mathbb{R}$ the $a = \sup A$ if $a \geq x$ for all $x \in A$ and $a' \geq x$ for all $x \in A$ then $x \leq y$. We define $\inf A$ be reversing the inequalities. If we allow $+\infty$ and $-\infty$ the $\sup A$ and $\inf A$ always exist.

Let $\{a_n\}$ be a sequence and define $i_n = \inf\{a_k : k \geq n\}$ and $s_n = \sup\{a_k : k \geq n\}$. Then

$$\liminf a_n = \lim i_n$$

and

$$\limsup a_n = \lim s_n.$$

If $x \neq 1$ then

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

***Continuity.*** Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a domain $D \subset \mathbb{R}$. Then

$$\lim_{x \to a} f = b$$

if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if for all $x \in D$ with $0 < |x - a| < \delta$ then $|f(x) - b| < \epsilon$. The function $f$ is *continuous* at $a$ if

$$\lim_{x \to a} f = f(a)$$

There is a theorem similar to Theorem 0.3 for limits of functions.

The function $f$ is *uniformly continuous* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $x, y \in D$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

**Theorem 0.4** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then there exits a $c$ and $d$ in $[a, b]$ such that $f(x) \leq f(c)$ and $f(x) \geq f(d)$ for all $x \in [a, b]$.*

**Theorem 0.5 (Intermediate Value Theorem)** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. If $y$ is between $f(a)$ and $f(b)$ then there exists a $x \in [a, b]$ such that $f(c) = y$.*

**Theorem 0.6** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then $f$ is uniformly continuous.*

A sequence of functions $f_n : D \longrightarrow \mathbb{R}$ *converges uniformly* to $f : D \longrightarrow \mathbb{R}$ if for all $\epsilon > 0$ there exists an $N > 0$ such that if $n > N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.

**Theorem 0.7** *Let $f_n : D \longrightarrow \mathbb{R}$ be continuous. If $f_n \to f$ uniformly then $f$ is continuous.*

***Derivatives.*** Define the *derivative* $f'(a)$ of the function $f$ at $a$ by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if it exists.

Differentiation rules (abbreviated):

1. $(f + g)'(a) = f'(a) + g'(a)$;
2. $(fg)(a) = f'(a)g(a) + f(a)g'(a)$;
3. $(f/g)(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$;
4. $(f \circ g)'(a) = f'(g(a))g'(a)$

**Theorem 0.8 (Mean Value Theorem)** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Theorem 0.9 (L'Hôpital's Rule)** *If $f(x), g(x) \to 0$ or $f(x), g(x) \to \infty$ as $x \to a$ then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

***Integrals.*** Let $P = \{x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$ and for $k = 1, \ldots, n$ set

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} \text{ and } m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}.$$

We then define the upper and lower sums for $P$ by

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

and

$$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

We define the upper and lower integrals by

$$\overline{\int_a^b} f(x)dx = \inf\{U(f,P) : P \text{ is a partition of } [0,1]\}$$

and

$$\underline{\int_a^b} f(x)dx = \sup\{L(f,P) : P \text{ is a partion of } [0,1]\}.$$

Then $f$ is integrable if $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$ and we write

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

**Theorem 0.10** *$f$ is integrable $\iff$ for all $\epsilon > 0$ there exist a partition $P$ such that $U(f,P) - L(f,P) < \epsilon \iff$ there exists partitions $P_n$ such that $U(f,P_n) - L(f,P_n) \to 0$.*

Properties of integrals (abbreviated):

1. $\int cf = c \int f$ if $c \in \mathbb{R}$;
2. $\int f + \int g = \int f + g$;
3. $|\int f| \leq \int |f|$;
4. $\int_a^b f(g(t))g'(t)dt = \int_{g(a)}^{g(b)} f(u)du$;
5. $\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$

**Theorem 0.11 (Fundamental Theorems of Calculus)**

*1.*

$$\int_a^b f'(x)dx = f(b) - f(a)$$

*2. Define*

$$F(x) = \int_a^x f(t)dt.$$

*If $f$ is continuous at $x$ then $F'(x) = f(x)$.*

***Series.*** Let $\{a_n\}$ be a sequence. Then the series $\sum_{k=0}^{\infty} a_k$ *converges* if the sequence of partial sums $s_n = \sum_{k=0}^{n} a_k$ converges. If $\sum_{k=0}^{\infty} |a_k|$ converges then the series $\sum_{k=0}^{\infty} a_k$ converges *absolutely.* If $\sum_{k=0}^{\infty} |a_k|$ doesn't converge but $\sum_{k=0}^{\infty} a_k$ does then the series converges *conditionally.*

Tests for convergence and divergence:

1. If $\sum_{k=0}^{\infty} a_n$ converges then $a_n \to 0$.
2. If $a_n \geq |b_n|$ and $\sum_{k=0}^{\infty} a_k$ converges then $\sum_{k=0}^{\infty} b_k$ converges absolutely.
3. Let $\{a_n\}$ be a sequence with $0 \leq a_{n+1} \leq a_n$ and let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a non-increasing function such that $f(n) = a_n$. Then $\sum_{k=1}^{\infty} a_k$ converges $\iff$

$$\int_1^{\infty} f(t)dt$$

converges. If $\sum_{k=1}^{\infty} a_k$ converges then

$$\int_1^{\infty} f(x)dx - a_1 \leq \sum_{k=1}^{\infty} a_k \leq \int_1^{\infty} f(x)dx.$$

4. Let $\rho = \limsup |a_n|^{1/n}$. Then $\sum_{k=0}^{\infty} a_k$ converges absolutely if $\rho < 1$ and diverges if $\rho > 1$.
5. Let $\rho = \lim |a_{n+1}|/|a_n|$ if it exists. Then $\sum_{k=0}^{\infty} a_k$ converges absolutely if $\rho < 1$ and diverges if $\rho > 1$.
6. Let $\{a_n\}$ be a sequence with $0 \leq a_{n+1} \leq a_n$. Then $\sum_{k=0}^{\infty} (-1)^k a_k$ converges $\iff a_n \to 0$.

Let $\sum_{k=0}^{\infty} c_k (x-a)^k$ be a power series and let

$$R = \frac{1}{\limsup |c_k|^{1/k}}.$$

Then the power series converges on any interval $(r - a, r + a)$ where $r < R$.

**Taylor's formula:** If

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

then

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some $c$ between $a$ and $x$.