Math 6510 - Homework 6
Due in class on 11/5/13

1. Let M_0, M_1 and N be differentiable manifolds and $f : M_0 \times M_1 \to N$ a smooth map. Then the map $F : M_0 \times TM_1 \to TN$ defined by $F(x_0, v) = (f_*(x_0, x_1))(0, v)$ is smooth where $v \in T_{x_1}M_1$. (You don't need to prove this but you should make sure that you know why its true!).

   Let G be a Lie group and define $f : G \times G \to G$ by $f(a, b) = ab$. Then use the above fact to show that a left-invariant vector field is smooth by showing that F restricted to $G \times \{v\}$ is an embedding where $v \in T_{\mathrm{id}}G$.

   Solution: We first show that $f_*(a, b)(v, w) = (R_b)_*(a)v + (L_a)_*(b)w$. To see this note that $T_{(a,b)}G \times G = T_aG \times T_bG$. The subspace $T_aG \times \{0\} \subset T_{(a,b)}G \times G$ is tangent to the sub manifold $G \times \{b\}$ and we have that $f|_{G\times\{b\}}(a) = R_b(a)$ so for $(v, 0) \in T_aG \times \{0\}$ we have $f_*(a, b)(v, 0) = (f|_{G\times\{b\}})_*(a)v = (R_b)_*(a)v$. Reversing the roles of a and b we similarly see that $f_*(a, b)(0, w) = (L_a)_*(b)w$ so $f_*(a, b)(v, w) = (R_b)_*(a)v + (L_a)_*(b)w$ as desired.

   Therefore we have that $F(a, v) = (L_a)_*(b)v$ where $v \in T_bG$. Recall that if $v \in T_{\mathrm{id}}G$ then $X_v(g) = (L_g)_*(\mathrm{id})v$ is the unique left-invariant vector field with $X_v(\mathrm{id}) = v$. Note that $X_v(g) = F(g, v)$. Since $G \times \{v\}$ is a smooth sub manifold of $G \times G$ the restriction of F to $G \times \{v\}$ is a smooth map from G to TG so the vector X_v is smooth.

2. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a one-dimensional sub-algebra. Show that for any left-invariant vector field $X \in \mathfrak{h}$ there is a flow ϕ_t defined for all $t \in \mathbb{R}$ with $\phi_t \in H \subset G$ where H is the Lie subgroup of G with Lie algebra $\mathfrak{h}$. Show that the map $t \mapsto \phi_t(\mathrm{id})$ is an onto homomorphism from the additive group $\mathbb{R}$ to H.

   Solution: Let $\phi_t : U \to G$ be the flow for X defined in a neighborhood U of $id \in G$ for $t \in (-\epsilon, \epsilon)$. Define a map $\Psi : G \times (-\epsilon, \epsilon) \to G \times G$ by $\Psi(g, t) = (g, \phi_t(\mathrm{id}))$ and the define $\Phi : G\times(-\epsilon, \epsilon) \to G$ by $\Phi = f\circ\Psi$ where f is the map from the previous problem. Then $\Psi_*(g, t)\frac{\partial}{\partial t} = (0, X(\phi_t(\mathrm{id}))) \in T_{(g,\phi_t(\mathrm{id}))}G \times G$ so by the chain rule $\Phi_*(g, t)\frac{\partial}{\partial t} = f_*(g, \phi_t(\mathrm{id}))\Psi_*(g, t)\frac{\partial}{\partial t} = (L_g)_*(\phi_t(\mathrm{id}))X(\phi_t(\mathrm{id})) = X(g)$ where we are using the calculation of f_* from the previous problem and the fact that X is left-invariant. Therefore Φ is a flow for X on all of G defined for $t \in (-\epsilon, \epsilon)$. Since Φ is defined on all of X for $t \in (-\epsilon, \epsilon)$ we then use our standard trick to extend Φ for all time $t \in \mathbb{R}$.

   The sub-algebra $\mathfrak{h}$ determines a 1-dimensional integral distribution on G and H is the leaf of the corresponding foliation that contains $id \in G$. The flow will preserve the leaves so $\phi_t(\mathrm{id}) \in H$. Since X is nowhere zero the map $t \mapsto \phi_t(\mathrm{id})$ will have injective derivative for all t and the map is locally injective and hence an open map. Let $h \in H$ be in the closure of the image of the map so there exists t_i with $\phi_{t_i}(\mathrm{id}) \to h$. Then for large i, the elements $h^{-1}\phi_{t_i}(\mathrm{id})$ are contained in the image since the map is locally injective. In particular, there exists s_i such that $\phi_{s_i}(\mathrm{id}) = h^{-1}\phi_{t_i}(\mathrm{id})$. But then $h = \phi_{t_i - s_i}(\mathrm{id})$ so h is in the image and the image is closed. In particular, the image is open and closed (in H) and is non-empty so it must be all of H.

3. Let G be a (connected) Lie group and $\mathfrak{g}$ its Lie algebra. Let $X \in \mathfrak{g}$ be a left-invariant vector field and $\phi_t \in G$ the associated flow. Show that $\mathrm{ad}_g X = X$ if and only if g commutes with ϕ_t. Conclude that ad_g is the identity on $\mathfrak{g}$ if and only if g is in the center of G.

Solution: The path $\phi_t(\text{id})$ has tangent X at in $T_{\text{id}}G$. If g commutes with ϕ_t then $\text{Ad}_g\,\phi_t(\text{id}) = \phi_t(\text{id})$ and therefore $\text{ad}_g X = X$.

For the other direction we view X as a left-invariant vector field on G. Then $\text{ad}_g X = (L_g)_*((R_{g^{-1}})_*X) = (R_{g^{-1}})_*((L_g)_*X) = (R_{g^{-1}})_*X$ since left and right-translation commute and X is left-invariant. If $\text{ad}_g X = X$ then $(R_{g^{-1}})_*X = X$ so $R_{g^{-1}}$ commutes with the flow ϕ_t for X. Note that $\phi_t(h) = L_h(\phi_t(\text{id})) = R_{\phi_t(\text{id})}(h)$ so $\phi_t = R_{\phi_t(\text{id})}$. So if $R_{g^{-1}}$ commutes with ϕ_t then g^{-1} (and therefore g) commutes with $\phi_t(\text{id})$.

By the above if g is in the center of G then ad_g acts as the identity on $\mathfrak{g}$. Conversely if ad_g acts as the identity on $\mathfrak{g}$ the for every $h \in G$ with $h = \phi_t(\text{id})$ where ϕ_t is the flow of some left-invariant vector field X we have that g commutes with h. By the next problem there is a neighborhood U of id in G such that U is in the image of the map exp so g commutes with everything in U. Note that the set of elements that commute with g is closed since the map from $G \times G \to G$ defined by $h \mapsto ghg^{-1}h^{-1}$ is continuous. To see that this set is open assume that h commutes with g. We then claim that the everything in the set $hU = \{f \in G | L_{h^{-1}}f \in U\}$ commutes with g. Since g and h commute we have $L_h Ad_g L_{h^{-1}} = Ad_g$ and therefore if $f \in hU$ we have $Ad_g f = L_h Ad_g L_{h^{-1}} f = L_h L_{h^{-1}} f = f$ where the second equality holds since $L_{h^{-1}} f \in U$ so g commutes with f. This proves that the set of elements that commute with g is also open and therefore must be all of the connected set G.

4. Define a map from $\mathfrak{g}$ to G as follows. For $X \in \mathfrak{g}$ let ϕ_t^X be the associated flow. Define $\exp(X) = \phi_1^X$. Note that $\mathfrak{g}$ is a vector space so $T_0\mathfrak{g} = \mathfrak{g}$ and $\exp_*(0) : \mathfrak{g} \to T_{\text{id}}G = \mathfrak{g}$. Show that $\exp_*(0) = id$.

   Solution: Let $\alpha : \mathbb{R} \to \mathfrak{g}$ be defined by $\alpha(s) = sX$. Then $\alpha'(0) = X$ so $\exp_*(0)X = (\exp \circ \alpha)'(0)$. Note that if ϕ_t^X is the flow for X then $\phi_t^{sX} = \phi_{st}^X$ so $\exp \circ \alpha(s) = \phi_1^{sX}(\text{id}) = \phi_s^X(\text{id})$ and by the definition of the flow the time $s = 0$ derivative of the path $\phi_s^X(\text{id})$ is X and therefore $\exp_*(0)X = (\exp \circ \alpha)'(0) = X$, as desired.

5. Let $G = GL(n)$ and recall that $\mathfrak{g} = M(n)$, the space of $n \times n$ matrices. Show that

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

   Solution: The series

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

   is really n^2 different series each of which will converge uniformly for X lying in a compact set in $M(n)$. If $X = (x_{ij})$ then let $\|X\| = \max_{i,j} |x_{ij}|$. Then via induction we see that $\|X^k\| \leq (n\|X\|)^k$. When X lies in a compact set the norm $\|X\|$ will be uniformly bounded and it follow that $\exp(X)$ converges uniformly on compact what sets in $M(n)$.

   Let $\bar{X}$ be the left-invariant vector field with $\bar{X}(\text{id}) = X$. In class we showed that $\bar{X}(A) = AX$. We claim that the flow for X on $GL(n)$ is $\phi_t = R_{\exp(tX)}$. To check this we need to calculate the time t tangent vector of the path $\alpha(t) = \phi_t(A) = A\exp(tX)$. But since the series converges uniformly on compact sets we can differentiate term by term to get $\alpha'(t) = A\exp(tX)X = \bar{X}(A\exp(tX)) = \bar{X}(\phi_t(A))$.