Math 6510 - Homework 7
Due in class on 11/26/13

1. The tangent bundle of a smooth manifold M is also a smooth manifold. Show that as a smooth manifold TM is orientable.

2. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ smooth. In class we showed that there are arbitrarily small $a \in \mathbb{R}^n$ such that all fixed points of $f_a(x) = f(x) + a$ are Lefschetz. Now let M be a smooth manifold, $f : M \to M$ a smooth map, $p \in M$ and U a neighborhood of p such that if $q \in U$ and $f(q) = q$ then $q = p$. Show that f is homotopic to a map g such that $f = g$ in the complement of U and g has only Lefschetz fixed points on U.

3. In proving the Poincare-Hopf Index theorem we used the following fact. Let $U_0, U_1 \subset \mathbb{R}^n$ be open and $f_0 : U_0 \to \mathbb{R}^n$ a smooth map that has an isolate fixed point at $p_0 \in U_0$. Let $G : U_0 \to U_1$ be a diffeomorphism. Then $f_1 = G \circ f_0 \circ G^{-1}$ has an isolated fixed point at $p_1 = G(p_0)$. Show that $L_{p_0}(f_0) = L_{p_1}(f_1)$. (Hint: In the definition for the local Lefschetz number at a point p we calculated the degree from a round sphere centered at p. Check that the definition works for any smooth sphere that bounds a manifold with p the only fixed point in the manifold.)

4. Let V_t be a one parameter family of vector fields on a compact manifold M. Define a vector field $\tilde{V}$ on $M \times \mathbb{R}$ by
$$\tilde{V}(x,t) = \left(V_t(x), \frac{\partial}{\partial t}\right).$$
Show that there exists a flow
$$\tilde{\Phi} : (M \times \mathbb{R}) \times \mathbb{R} \longrightarrow M \times \mathbb{R}$$
defined for all time (even though $M \times \mathbb{R}$ is not compact). Let
$$\Phi : M \times \mathbb{R} \longrightarrow M$$
be defined by $\Phi(x,t) = \pi(\tilde{\Phi}(x,0,t))$ where $\pi : M \times \mathbb{R} \longrightarrow M$ is the projection of Y onto its first factor. Show that
$$\Phi_*(x,t)\frac{\partial}{\partial t} = V_t(\Phi(x,t)).$$
In this case Φ is the flow of the *time dependent* vector field v_t.