Math 6510 - Homework 5

Due in class on 10/22/13

Recall that $O(n)$ is the group of $n \times n$ matrices A with $AA^T = I$ and that it is a differentiable manifold. Let $G(n) = \mathbb{R}^n \ltimes O(n)$ where $(v_0, A_0) \cdot (v_1, A_1) = (v_0 + A_0 v_1, A_0 A_1)$. For $T = (v, A) \in G$ define (in abuse of notation) $T : \mathbb{R}^n \to \mathbb{R}^n$ by $Tx = Ax + v$.

1. If $T_0, T_1 \in G(n)$ then we can multiply them as elements of $G(n)$ and compose them as maps of $\mathbb{R}^n$. Show that $T_1 \cdot T_0 = T_1 \circ T_0$.

2. A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry if $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbb{R}^n$. Show that every isometry of $\mathbb{R}^n$ is represented by an element in $G(n)$.

   (a) Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry. Show that there exist a $T_0 \in G$ such that $T_0 \circ S(0) = 0$.

   (b) Let $e_i = (0, \dots, 1, \dots 0) \in \mathbb{R}^n$ where the 1 is in the ith place. Find a $T_1 \in G$ such that $T_1 \circ T_0 \circ S(e_i) = e_i$ and $T_1 \circ T_0 \circ S(0) = 0$.

   (c) Show that $T_1 \circ T_0 \circ S(x) = x$ and therefore $S = (T_1 T_0)^{-1}$.

3. Let $U \subset \mathbb{R}^n$ be open and connected and $\phi : U \to \mathbb{R}^n$ an isometry onto its image. Show that ϕ is the restriction of an element in $G(n)$.

4. For all $x \in \mathbb{R}^n$ the tangent space $T_x\mathbb{R}^n$ can be canonically identified with $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Show that f is an isometry if for all $x \in \mathbb{R}^n$ and $v \in T_x\mathbb{R}^n$ then $|f_*(x)v| = |v|$. In particular, if $f_*(x) \in O(n)$ for all $x \in \mathbb{R}^n$ show that f is an isometry and conclude that $f_*(x) \equiv A$ for some $A \in O(n)$.

5. Let T_s be a smooth path in G with $T_0 = I$. For each $x \in \mathbb{R}^n$, $\alpha_x(s) = T_s(x)$ is a smooth path. Let $V(x) = \alpha_x'(0)$. Then $V(x)$ is a vector field on $\mathbb{R}^n$. Show that $V(x) = Ax + v$ where A is a skew-symmetric $n \times n$ matrix and $v \in \mathbb{R}^n$.

6. Given a vector field $V(x) = Ax + v$ of the above form show that there exists a flow ϕ_t for V defined on all of $\mathbb{R}^n$ and for all time t. Further show that $\phi_t \in G(n)$. Here is one way to to do this. Let $U \subset \mathbb{R}^n$ an open set with compact closure. Then ϕ_t exists for $t \in (-\epsilon, \epsilon)$. We'll show that ϕ_t is the restriction to U of a path in $G(n)$.

   For $x \in U$, let $v \in T_xU$ and let $h_v(t) = |(\phi_t)_*(x)v|^2$. Since $\phi_0(x) = x$ we have that $h_v(0) = |v|^2$. We want to show that h_v is constant and then ϕ_t is an isometry by (4).

   (a) We first calculate $h_v'(0)$. Let $B(t) = (\phi_t)_*(x)$. Show that
   $$h_v(t) = v^T B(t)^T B(t) v.$$

   (b) Let $\dot{B}$ be the derivative of $B(t)$ at $t = 0$. Using the fact that we can write $\phi_t(x) = x + t\psi_t(x)$ show that $\dot{B} = A$. Conclude that
   $$h_v'(0) = v^T(\dot{B}^T B(0) + B(0)^T \dot{B})v = v^T(A^T I + IA)v = 0.$$

   (c) To calculate $h_v'(s)$ we replace U with $W = \phi_s(U)$, x with $y = \phi_s(x)$, v with $w = (\phi_s)_*(x)v$ and the flow with $\phi_s \circ \phi_t \circ \phi_s^{-1}$. (Note that this last composition changes the domain of the flow. Where U and W intersect the two flows are equal.) We can then define h_w as above. Show that $h_v'(t) = h_w'(0)$ and conclude that $h_v' \equiv 0$ and therefore ϕ_t is an isometry.

(d) Show that ϕ_t can be extended to a flow of V on all of $\mathbb{R}^n$ for all time.

7. Note that T_s is a smooth path in $G(n)$ so its derivative at $s = 0$ determines a tangent vector $\dot{T}$ in $T_I(G(n))$. Use (4) and (5) to show that the vector field V is determined by $\dot{T}$.

8. Let $\mathfrak{g}(n)$ be all vector fields of the above form. Show that $\mathfrak{g}(n)$ is a vector space of dimension $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ and that the natural map from $T_I G(n)$ to $\mathfrak{g}(n)$ is an isomorphism.

9. Let $S = (w, B) \in G(n)$. Define map $ad_B : \mathfrak{g}(n) \to \mathfrak{g}(n)$ as follows. Given $V \in \mathfrak{g}(n)$ there exists a path T_s in $G(n)$ whose derivative when $s = 0$ is V. Let $\tilde{T}_s = ST_sS^{-1}$ and let $ad_B(V)$ be the time zero derivative of this path. Show that ad_B is well defined and linear. In particular if $V(x) = Ax + v$ show that

$$ad_B(V)(x) = BAB^{-1}x - BAB^{-1}w + Bv.$$

10. Now let $n = 2$ and define a basis for $\mathfrak{g}(2)$ by $V_1(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$, $V_2(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $V_3(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Using (8) show that in this basis

$$ad_B = \begin{pmatrix} \det B & 0 \\ -\det B w^{\perp} & B \end{pmatrix}$$

where $w^{\perp} = V_1(w)$ is a $\pi/2$-counter clockwise rotation of w. Note that this is a 3×3 matrix and in this way $G(2)$ can be represented as a group of matrices. (With more work we could do this for any $G(n)$.)