**Final**

Due at 4 PM on 12/20/13

Do as many problems as you can in 1 1/2 hours.

1. Let $B$ be an $n \times n$-diagonal matrix with all diagonal entries 1 except the last one which is $-1$. Let $O(n-1,1)$ be the set of $n \times n$-matrices $A$ with $ABA^T = B$. Show that $O(n-1,1)$ is a sub-manifold of $R^{n^2}$. For $O(1,1)$ calculate the tangent space at the identity.

2. Let $U \subset \mathbb{R}^n$ be a connected open set that with $p, q \in U$. Show that there exists a diffeomorphism $\phi$ of $\mathbb{R}^n$ such that $\phi(p) = q$ and $\phi$ is the identity outside $U$. (Hint: First assume that $U$ is convex with compact closure and find a vector field with support in $U$ whose flow takes $p$ to $q$.)

3. Show that the tangent bundle of $S^2$ is not trivial.

4. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (x^2 - y^2, 2xy)$ and calculate $f^* dx$.

5. Let $X$ and $Y$ be manifolds and $f : X \to \mathbb{R}^n$ and $g : Y \to \mathbb{R}^n$ be smooth maps and let $F : X \times Y \to \mathbb{R}^n$ be defined by $F(x,y) = f(x) - g(y)$. Calculate the tangent map $F_*$ in terms of $f_*$ and $g_*$. Make sure to justify your calculation. Let $f_a(x) = x + a$ for $a \in \mathbb{R}^n$. Show that for almost every $a \in \mathbb{R}^n$, $f_a$ and $g$ are transverse.

6. Let $S^2 = \{(x,y,z) | x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $N \in \Lambda(\mathbb{R}^3)$ be the vector field on $\mathbb{R}^3$ defined by $N(x,y,z) = (x,y,z)$. For any vectors $V, W \in T_pS^2$ the column vectors $N(p), V$ and $W$ determine a matrix. Define a 2-form $\omega \in \Omega^2(S^2)$ by setting $\omega(p)(V,W)$ to be the determinate of this matrix. Show that $S^2$ has an orientation such that for every oriented basis $\{V, W\}$ of $T_pS^2$, $\omega(p)(V,W) > 0$. Use this to show that $\int_{S^2} \omega > 0$ and that $\omega$ is not exact.