Math 6510 - Homework 9

Due at 4 PM on 11/24/04

1. Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x, y) = (1, 3y^{2/3})$. Let $\alpha_1(t) = (t, 0)$ and $\alpha_2(t) = (t, t^3)$. Show that for $i = 1, 2$

$$\begin{aligned} \alpha_i'(t) &= f(\alpha_i(t)) \\ \alpha_i(0) &= (0, 0) \end{aligned}$$

   Why does this not contradict the uniqueness for ODE's that we proved in class?

2. Let v_t be a one parameter family of vector fields on a compact manifold X. Define a vector field w on $Y = X \times \mathbb{R}$ by

$$w(x, t) = \left(v_t(x), \frac{\partial}{\partial t} \right).$$

   Show that there exists a flow

$$\Psi : Y \times \mathbb{R} \longrightarrow Y$$

   defined for all time (even though Y is not compact). Let

$$\Phi : X \times \mathbb{R} \longrightarrow X$$

   be defined by $\Phi(x, t) = \pi(\Psi(x, t, t))$ where $\pi : Y \longrightarrow X$ is the projection of Y onto its first factor. Set $\alpha_x(t) = \Phi(x, t)$ and show that

$$\begin{aligned} (d\alpha_x)_{t_0} \frac{\partial}{\partial t} &= v_{t_0}(\alpha_x(t_0)) \\ \alpha_x(0) &= x \end{aligned}$$

   In this case Φ is the flow of the *time dependent* vector field v_t.

3. Let X be a manifold, $U \subset X$ an open subset and

$$\Phi : U \times (-b, b) \longrightarrow X$$

   the flow of a vector field v on X. Let $\phi_t(x) = \Phi(x, t)$. Show that

$$\phi_{t+s} = \phi_t \circ \phi_s$$

   where defined. (I did this in class in a slapdash way. You should write down a more careful proof.)