2024

1. What is the smallest positive integer n such that given any set of n natural numbers no greater than 2024 there must exist two of them whose ratio lies in the interval $(1, 2]$?
2. Let n be a positive integer. Show that any sequence of more than n^2 points in an equilateral triangle with side length 1 must contain 2 points whose distance is less than or equal to $\frac{1}{n}$.
3. A die is a cube with faces numbered 1 through 6. A *loaded die* is a die in which the faces do not necessarily appear with equal probability when the die is rolled. It is allowed that a face has a zero probability of appearing. Show that given two loaded dice (not necessarily loaded in the same way) it cannot happen that when rolling the pair of dice arbitrarily many times that each of the possible sums $2, \cdots, 12$ appears with the same probability in the limit.
4. Show that for any square matrix A there is some positive integer k so that
$$I + A + A^2 + \cdots + A^k$$
is invertible, where I is the identity matrix.
5. Let $\{z_1, \cdots, z_n\}$ be a finite sequence of complex numbers, $n \geq 2$, each of which has real part equal to 1. Prove that
$$\sum_{1\leq j<k\leq n} z_j z_k \neq 0$$
6. Let A and B be two $n \times n$ matrices with entries in $\mathbb{C}$ and r, s be two positive integers. Prove that if all the eigenvalues of A are even integers and $A^r B - BA^s = B$, then B is a zero matrix.
7. Prove that the integral $\int_0^{2\pi} \ln|e^{ix} - 1|dx$ converges and compute its value.

2023

1. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a monic polynomial with integer coefficients. Suppose that $P(0) = 2023$ and that for every irrational number x the number $P(x)$ is also irrational. Prove that the polynomial is $P(x) = x + 2023$.
2. Find the prime factorization of the sum $S = 2 \times 2^2 + 3 \times 2^3 + \cdots + 2023 \times 2^{2023}$.
3. Let m and n be distinct positive integers. Prove that
$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!n!}{m^m n^n + m^n n^m}$$
4. Let A, B, C be $n \times n$ matrices with complex entries such that $A + B = AB$, $B + C = BC$ and $C + A = CA$. Prove that the only possible eigenvalues of A are 0 or 2.
5. Suppose that M is a 3×3 matrix with determinant 2023 and integer eigenvalues. Let i, j, k be the entries of M on the main diagonal, and suppose that i, j, k are distinct, single-digit positive or negative integers. Prove that M is diagonalizable.
6. There are 6 frogs in the plane, situated at the vertices of a regular hexagon. The frogs are allowed to move one at a time in the following way. A frog at a point A is allowed to jump over a frog at a point B and land at the point C satisfying $C - A = 3(B - A)$. Show that no matter how the frogs jump, it is impossible for a frog to land at the center of the hexagon.
7. Simplify $\lim_{n\to\infty} \sum_{k=0}^{n-1} (\frac{n^2}{n^2+k^2} - \frac{\pi}{4})$.

2022

1. Let p be a prime number. Prove that $p - 8$ is not the cube of a positive integer.
2. Compute the convergent of the following limit as a real number if it exists. If it doesn't exist explain why.
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i}$$

3. For positive integers n, a, b, c with $2a > b$, evaluate
$$\lim_{n\to\infty} \left\{ \sqrt{a^2n^2 + b \cdot n + c} \right\},$$
where $\{x\}$ denotes the fractional part of x, that is, $\{x\}$ is the least non-negative real number such that $x - \{x\}$ is an integer.
4. Alice and Bob are playing the following game:

   > There are 100 coins of various denominations lined up in a row on the table. Alice starts by taking either the first or last coin. Then Bob takes either the first or the last coin from remaining row of 99 coins. They alternate taking one coin from either end of the remaining row of coins until all coins are taken.

   Show that Alice can ensure that she gets at least as much money as Bob.
5. Let $f(x)$ be a function such that:
   (a) The lowest point on the parabola $y = f(x)$ has $y = -3$, and
   (b) A local maximum for $y = f(f(x))$ occurs at $\left(-\frac{1}{2},\ 22\right)$.

   Prove that $y = f(f(f(x)))$ has a critical point with $y = 2022$.
6. Consider three equilateral triangles OAB, OCD, OEF with a common vertex in the plane, where the label of each triangle lists the vertices in a counterclockwise order. Prove that the midpoints of the segments BC, DE and FA are the vertices of an equilateral triangle.
7. Assume that f is a nonzero continuous function on the interval $[0, 1]$. Define
$$a_n = \int_0^1 [f(x)]^{2n}\, dx \qquad \text{for } n = 1, 2, 3, \ldots.$$
Prove that the sequence $\left(\dfrac{a_{n+1}}{a_n}\right)$ is convergent.

2021

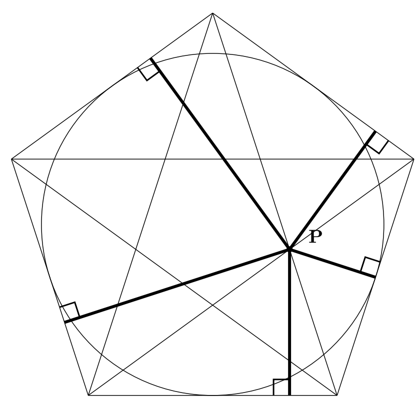
1. Prove that $5^{2^n} \equiv 1 \bmod (2^{n+2})$ for all non-negative integers n.
2. Determine the continuous functions $f(x)$ with real values on the interval $[0, 1]$ satisfying
$$\int_0^1 x f(x)\, dx \geq \frac{1}{3} \geq \int_0^1 f^2(x)\, dx.$$
3. Find all real-valued solutions to the equation $\lfloor x \rfloor = x^4 - 2x^2$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.
4. Let n be an odd integer greater than 1. Let A be a symmetric $n \times n$ matrix with the property that each row and each column contains each of the integers $1, \ldots, n$. Prove that each of the integers $1, \ldots, n$ appears on the main diagonal.
5. Given $n \geq 2$ complex numbers $z_1, z_2, \ldots, z_n$, with $z_1 + z_2 + \cdots + z_n = 0$ and $|z_i| = 1$ for each i, prove that for any complex number z we have $\sum_{i=1}^n |z - z_i| \geq n$.
6. Does there exist a positive sequence a_n such that $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty \frac{1}{n^2 a_n}$ are both convergent?

2019

1. Let $x_1, x_2, \ldots, x_{2019}$ be positive integers such that
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{2019}} \geq 12.$$
Prove that at least two of these integers must be equal.

2. Let A be an $n \times n$ matrix with the property that replacing any single row of A with a row of all 1's results in a matrix which is not invertible. Show that A is not invertible.
3. Let n be a positive integer, and let $\mathrm{LCM}(1, 2, \ldots, 2n+1)$ denote the least common multiple of $1,\ 2,\ \ldots,\ 2n+1$.
   a) Show that
   $$\mathrm{LCM}(1, 2, \ldots, 2n+1) \int_0^1 x^n (1-x)^n \,\mathrm{d}\,x$$
   is an integer.

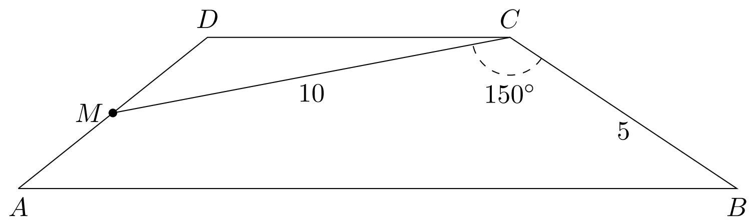
   b) Deduce that $\mathrm{LCM}(1, 2, \ldots, 2n+1) > 4^n$.
4. Find all positive integers n for which
   $$3007^n - 112^n - 315^n + 1458^n$$
   is divisible by 2019.
5. Suppose a regular pentagon is circumscribed around a circle of radius 1, and a pentagram (5-pointed star) inscribed within it. Let **P** be one of the inner vertices of the pentagram (as in the diagram below). Find the sum of the five distances from **P** to each of the five sides of the outer pentagon.



6. Let a, b and c be real numbers such that $a^2 - 3b \leq \frac{3}{4}$. If the polynomial
   $$P(x) = x^3 + ax^2 + bx + c$$
   has real roots $x_1 \leq x_2 \leq x_3$, prove that $x_3 - x_1 \leq 1$.
7. If $\alpha,\ \beta$, and γ are the angles of an arbitrary triangle, show that
   $$\sin \tfrac{\alpha}{2} \sin \tfrac{\beta}{2} \sin \tfrac{\gamma}{2} < \tfrac{1}{4}.$$

2018

1. Let $ABCD$ be a trapezoid, with parallel sides AB and CD. Let M be the midpoint of side AD. Assume that the angle BCM is 150 degrees. Suppose that the length of BC is 5 and the length of CM is 10. Find the area of the trapezoid $ABCD$. (See the diagram below, which is not to scale.)



2. Find all solutions to
$$(m^2+n)(m+n^2) = (m+n)^3$$
with $m, n \in \mathbb{Z}$.
3. Let $N > 1$ be an integer and let k be an integer with $1 \leq k \leq N-1$. Prove that
$$\sum_{n=0}^{N-1} \cos\left(\frac{2\pi kn}{N}\right) = 0.$$
4. Let $n \geq 1$ be an integer. If $\lfloor x \rfloor$ represents the largest integer less than or equal to x, prove that
$$\lfloor \sqrt{n} + \sqrt{n+3} \rfloor = \lfloor \sqrt{4n+6} \rfloor.$$
5. Let $f(x) = 4^x + 6^x + 9^x$. Prove that if m and n are positive integers with $m \leq n$, then $f(2^m)$ divides $f(2^n)$.
6. A closed knight's tour on an $m \times n$ chessboard is a sequence of mn knight's moves such that the knight starts in a specific square, visits every other square exactly once, and ends the last move in the original starting square.

   Prove that there is no closed knight's tour on a $4 \times n$ chessboard, for any value of $n \geq 1$.
7. Prove that for any positive integer n,
$$\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

2017

1. Set $x_0 = 1$, and for each integer $n \geq 0$ recursively define $x_{n+1} = \log_3(3^{x_n} - x_n)$. Determine
$$\sum_{n=0}^{\infty} x_n.$$
2. Let A and B be $n \times n$ matrices with real entries, and let I_n be the $n \times n$ identity matrix. Show that if $I_n - AB$ is invertible, then so is $I_n - BA$.
3. Find all pairs of positive integer solutions (x, y) to the equation
$$x^2 + y^2 = 2017(x-y).$$
4. Given fifty-one points in a closed square of side length 1, show that there are (at least) three points covered by a single closed disc of radius $1/7$.
5. The sum of ten distinct positive integers is 62. Prove that their product is divisible by 720.
6. Let $n \geq 5$ be an integer, and suppose n points are given in the Euclidean plane no three of which lie on a line. Karin and Mary play the following turn-based game: On a player's turn she draws a line segment between any two points which have not previously been connected by a line segment. The winner is the one after whose move every point is at the end of at least one line segment. If Karin is the first player, find the values of n for which she has a winning strategy, irrespective of how Mary plays.

7. Let $f : [0,1] \to \mathbb{R}$ be a function which is differentiable. (To be differentiable at the endpoint $x = 0$ means that the limit $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$ exists. A similar statement holds for the other endpoint $x = 1$.) Suppose that f and f' have no common zero. Show that the set of zeroes of f in $[0,1]$ is finite.

2016

1. Find all functions $f : \mathbb{R} - \{1/3, -1/3\} \to \mathbb{R}$, satisfying for all x in the domain the equality
$$f\left(\frac{x+1}{1-3x}\right) + f(x) = x.$$
2. Find the limit of the sequence $a_0, a_1, a_2, \dots$ given by $a_0 = 1$, $a_1 = 2$, a nd
$$na_n + a_{n-2} = (n+1)a_{n-1} \text{ for } n \geq 2.$$
3. Prove that there exists a square integer which can be written simultaneously as the sum of 2, 3, 4, and 5 non-zero perfect squares.
4. Let C be a circle with center M and radius r. Let C' be a different cir cle, with center M'. Further suppose that C and C' intersect at two disti nct points P and P'. Let d be the distance from M to M'. Suppose tha t the two tangents to C and C' at P intersect perpendicularly. Let S be the intersection of the segments $\overline{MM'}$ and $\overline{PP'}$. Find t he length of the segment $\overline{MS}$ as a function of d and r.
5. On a line segment choose randomly two distinct points (not at the ends of the se gment). What is the probability that the three resulting segments can form a tr iangle?
6. Show that if $1^1 + 2^2 + 3^3 + 4^4 + \cdots + 2016^{2016} = m^k$ with $m, k \in \mathbb{Z}$, then $k < 3$.
7. Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers, such that for a ll $n \geq 1$
$$a_{n+1} \leq a_n + \frac{1}{(n+1)^2}.$$
Prove that the sequence is convergent.

2015

1. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that the following two conditions hold:
   (i) For all $n \in \mathbb{Z}$ we have $f(n)f(-n) = f(n^2)$.
   (ii) For all $m, n \in \mathbb{Z}$ we have $f(m+n) = f(m) + f(n) + 2mn$.
2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ||x| - 1|$. Find all solutions $x \in \mathbb{R}$ to
$$(\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}})(x) = x$$
with n a positive integer. (Note: The answer may depend on n.)
3. Find all pairs of nonnegative integers x, y such that
$$\sqrt{x^2 + y + 1} + \sqrt{y^2 + x + 4}$$
is an integer.
4. The two tangent lines to a circle C at points $P \neq Q$ intersect at a point A, and similarly the two tangent lines to C at points $P' \neq Q'$ intersect at a point A'. If A' is on the line generated by PQ, prove that A is on the line generated by $P'Q'$.
5. Let $A \subset \mathbb{R}$ be a finite, non-empty set of real numbers, and let $f : A \to A$ be a function. Assume for every $x, y \in A$ with $x \neq y$, it happens that $|f(x) - f(y)| < |x - y|$. Prove there exists some $a \in A$ such that $f(a) = a$.
6. Find all polynomials with real coefficients $P(x) \in \mathbb{R}[x]$ satisfying:
$$(x+1)^3P(x-1) - (x-1)^3P(x+1) = 4(x^2-1)P(x).$$

7. Determine

$$\lim_{n\to\infty} n^2 \left[\left(1+\frac{1}{n+1}\right)^{n+1} - \left(1+\frac{1}{n}\right)^{n} \right].$$

2014

1. Given that $f:\mathbb{R}\to\mathbb{R}$ is continuous and $\int_0^1 f(x)\,dx = a$, evaluate

$$\int_0^1 f(x) \left(\int_0^x f(t)\,dt \right) \left(\int_x^1 f(t)\,dt \right) dx.$$

2. Let $ABCD$ be a square, with side length 1. On sides CD and AD are points P and Q (respectively) such that the perimeter of the triangle PDQ is 2. Show that the angle PBQ is $45°$.
3. Consider six points in the plane, no three of which are on any given line. Thus, they determine fifteen segments and twenty triangles. If all the segments have different lengths, prove that there is a segment which is the smallest side of a triangle and the largest side of another triangle.
4. Determine

$$\lim_{n\to\infty} \left(1+\frac{1}{\ln(n)}\right)\left(1+\frac{1}{2\ln(n)}\right)\cdots\left(1+\frac{1}{n\ln(n)}\right).$$

5. Evaluate

$$\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}\,dx.$$

6. Given integers $n\ge 3$ and $1\le i<j\le n-1$, prove that the binomial coefficients $\binom{n}{i}$ and $\binom{n}{j}$ are not relatively prime.
7. Let $f:[0,1]\to(0,\infty)$ be a continuous function satisfying

$$\int_0^1 f(x)\,dx \int_0^1 \frac{1}{f(x)}\,dx = 1.$$

Show that f is constant.

2013

1. Five boys and five girls sit around a table. Prove that there is someone sitting between two girls.
2. Let X, Y be two $n\times n$ matrices such that $XY = X+Y$. Prove that $XY = YX$.
3. A 7×7 square is tiled with ten 4×1 rectangles and one 3×3 square. What are the possible positions of the 3×3 square?
4. Let $A=(a_{ij})$ be an $n\times n$ matrix. Let $B_1,\ldots,B_n$ be $m\times m$ matrices. Let C be a block matrix, consisting of n^2 blocks $a_{ij}B_j$:

$$C = \begin{bmatrix} a_{11}B_1 & a_{12}B_2 & \cdots & a_{1n}B_n \\ a_{21}B_1 & a_{22}B_2 & \cdots & a_{2n}B_n \\ \vdots & \vdots & & \vdots \\ a_{n1}B_1 & a_{n2}B_2 & \cdots & a_{nn}B_n \end{bmatrix}$$

Express the determinant of C in terms of the determinants of A and $B_1,\ldots,B_n$.
5. Let $a_1,a_2,\cdots,a_{2n},b_1,b_2,\cdots,b_{2n}$ be non-negative real numbers such that $a_1=a_{2n}$ and $b_1=b_{2n}$. Prove that

$$min_i(a_i+b_i) \le \sum_{i=1}^{2n-1} min\{a_i,a_{i+1}\} + \sum_{i=1}^{2n-1} min\{b_i,b_{i+1}\}$$

2012

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y, z the relation
$$f(f(x+y)+z) + f(x + f(y+z)) = 2y$$
holds.
2. Triangle ABC has side lengths a, b, and c and median lengths α, β, and γ. If α, β, γ are the side lengths of a second triangle, what are the median lengths in that triangle?
3. Determine all the real solutions of the equation
$$\left(x^3 + \frac{3}{4}x\right)^3 + \frac{3}{4}\left(x^3 + \frac{3}{4}x\right) = x.$$
4. Prove that for every polynomial P there is a polynomial Q such that $Q(x^{2012})$ is a multiple of $P(x)$.
5. Find all pairs (m, n) of positive integers for which $4^m + 5^n$ is a square.
6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $f(0) = 0$ and $f(1) = 1$. Prove that
$$\frac{1}{e} \le \int_0^1 |f'(x) - 2xf(x)|\, dx.$$
7. For all real numbers a, b, c consider the inequality
$$|a-b| + |b-c| + |c-a| \le C\sqrt{a^2+b^2+c^2}.$$
   (a) Prove the inequality for $C = 2\sqrt{2}$.
   (b) Prove that under the additional assumption $a, b, c \ge 0$ the inequality also holds for $C = 2$.

2011

1. Evaluate the sum of the series
$$\sum_{k=1}^{\infty} \frac{1}{2k^2 + k}.$$
2. Prove that if m, n are positive integers such that $\sqrt{7} > \frac{m}{n}$, then $\sqrt{7} > \frac{m}{n} + \frac{1}{mn}$.
3. Solve the equation
$$x^2 + xy + y^2 = 97$$
for (i) natural numbers x, y, and (ii) integer numbers x, y.
4. Prove that for any positive integers a, b, c, d the product $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ is divisible by 12.
5. Which positive integers can be written as the sum of ≥ 2 consecutive positive integers?
6. Let $a > 0$ be a constant. Assume that $x_0 > 0$ and
$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right).$$
Prove that $\lim_{n\to\infty} x_n$ exists and find it.
7. Let $x_1, x_2, \ldots, x_n \ge 1$. Prove that
$$(1 + \sqrt[n]{x_1 x_2 \cdots x_n})\left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n}\right) \ge n.$$

2010

1. Let S be a square. Prove that S can be divided into n squares, using line segments parallel or perpendicular to the sides of S, for each integer $n \ge 6$.
2. Evaluate the following if it exists:
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{nk}}.$$

3. Let ΔABC be an arbitrary triangle in $\mathbb{R}^2$ with vertices A, B, and C. A frog starts from a point $P_0 \in \mathbb{R}^2$ and travels directly toward A. Upon reaching A the frog continues in the same direction to the point P_1 such that $P_0A = AP_1$. Next the frog travels from P_1 directly through B to the point P_2 such that $P_1B = BP_2$. The frog then starts from P_2 and travels through C to the point P_3 such that $P_2C = CP_3$. Next from P_3, the frog repeats the same action with respect to A, B, and C cyclicly, generating a sequence of points $P_1, P_2, P_3, P_4, \ldots$. What is the distance between P_0 and P_{2010}?
4. Define a sequence recursively by $x_1 = 1$, $x_2 = 1$, and $x_n = 3x_{n-1} - x_{n-2}$ for $n > 2$. Find a closed formula for x_n.
5. Prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n + 1$ for all $n \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
6. For which positive integers n can the $n \times n$ chess board with a corner square removed be tiled by 3×1 dominoes? For which n can the $n \times n$ chess board with some square removed be tiled by 3×1 dominoes?
7. A number of students sit in a circle while their teacher give them candy. Each student initially has an even number of pieces of candy. When the teacher blows a whistle, each student simultaneously gives half of his or her own candy to the neighbor on the right. Any student who ends up with an odd number of pieces of candy gets one more piece from the teacher. Show that no matter what the distribution is at the beginning, after a finite number of iterations of this transformation all students will have the same number of pieces of candy.

2009

1. Let S and S' be unit squares in $\mathbb{R}^2$ with their centers at the origin. Find the minimum area of their intersection $S \bigcap S'$. (see Fig. 1)
2. Let $n = 3k + 1$ with $k = 1, 2, 3, \cdots$. Consider the $n \times n$ chess board. How many of the n^2 squares S have the property that after S is removed the remaining $n^2 - 1$ squares can be tiled by 3×1 dominoes?
3. Let $x, y, z \in \mathbb{R}$ satisfy $x^2 + y^2 + z^2 = 1$ and $x + y + z = 0$. Find $\max(xyz)$ and $\min(xyz)$.
4. Find an explicit real valued function $f : \mathbb{R} \to \mathbb{R}$ (in closed form) whose Taylor series equals
$$f(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots$$
5. Prove that
$$\lim_{n\to\infty} \frac{n}{2^n} \sum_{k=1}^{n} \frac{2^k}{k} = 2$$
6. Let $x_1 < x_2 < x_3 < \cdots < x_n$ be n real numbers, where integer $n > 1$. Prove that
$$\sum_{i=1}^{n} \frac{1}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (x_i - x_j)} = \sum_{i=1}^{n} \frac{1}{(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)} = 0.$$
7. Let $f(x) : \mathbb{R}^1 \to [0, \infty)$ be continuous and differentiable. Prove
$$\int_0^t \int_0^t f(xy)dxdy + \int_0^t \int_0^t xyf'(xy)dxdy = \int_0^{t^2} f(s)ds.$$