OUTER LIMITS

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Abstract. Let $\overline{\mathcal{Y}}_n$ denote compactified Outer Space of rank n. An F_n-tree represents a point of $\overline{\mathcal{Y}}_n$ if and only if it is very small. There exist nongeometric very small (even free) actions that arise as the attracting fixed points of irreducible outer automorphisms of F_n. The dimension of $\overline{\mathcal{Y}}_n$ is $3n-4$

0. Introduction

Topologists study groups by analyzing spaces on which they act. Interesting compactifications of such spaces often lead to further group-theoretic information. The model case arises when a group G acts on a contractible space X compactified by ∂X to $\overline{X}$ satisfying the following.

(1) The action of G on X is free, properly discontinuous, and cocompact.
(2) The action extends to $\overline{X}$.
(3) $\overline{X}$ is finite dimensional.
(4) $\overline{X}$ is an absolute retract.
(5) ∂X is a Z-set in $\overline{X}$.
(6) The set of translates of a fundamental domain in X forms a null-sequence in $\overline{X}$.

Some consequences of these properties are:

- (4) implies $cd(G) \le dim(\partial X) + 1$,
- (1)-(6) imply $cd(G) = dim(\partial X) + 1$ [BM],
- (1), (3)-(6) imply the Novikov Conjecture for G [FW].

Also, under the conditions (1), (2), (6), and the contractibility of $\overline{X}$, Carlsson and Pedersen have some results about splitting the assembly maps [CP]. An example of a situation where (1)-(6) hold is a torsion-free hyperbolic group acting on its Rips complex compactified with the Gromov boundary [Gr] [BM].

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Many times a natural compactification of a G-space does not satisfy all of the above properties. For example, $SL_n(\mathbb{Z})$ (and many other arithmetic groups) acts on its symmetric space which compactifies to a ball; the action satisfies all of the above properties, except that it is neither free nor cocompact. Another example is Mapping Class Group acting on Teichmüller Space compactified by the space of projectivized measured laminations.

It is from this point of view that in this paper we study the action of $Out(F_n)$ on the closure $\overline{\mathcal{Y}}_n$ of rank n Outer Space $\mathcal{Y}_n$ in the projectivized space of nontrivial F_n-trees. (Recall that outer space of rank n may be identified with the subset of the space of projectivized F_n-trees corresponding to free simplicial actions.) Freeness clearly fails since the group has torsion. However, the point stabilizers are finite, which is often adequate in applications. Moreover, the action is not cocompact, but the quotient space is a finite union of open simplices, a property analogous to finite volume and replacing cocompactness. Property (2) clearly holds. Property (6) fails in rank 2 [CV2]. Since $Out(F_2)$ embeds in $Out(F_n)$ and $\overline{\mathcal{Y}}_2$ equivariantly embeds in $\overline{\mathcal{Y}}_n$, Property (6) fails. Property (5) fails for similar reasons. We hope that the extent of the failure of Properties (5) and (6) may be understood.

After the preliminary results of Section 1, Section 2 is devoted to a main result of this paper; the characterization of $\mathbb{R}$-trees representing points in $\overline{\mathcal{Y}}_n$; these are precisely the very small actions of Cohen-Lustig[CL]. The key is to use the techniques of Rips, further developed in [BF1], to understand geometric actions (roughly, those actions that are dual to foliated band complexes, see Definition 1.1.) We show that geometric actions split into simpler actions, and argue inductively.

In section 3, we give an algorithm for deciding whether or not the attracting fixed point of a given exponentially growing irreducible outer automorphism is geoemtric. This section is independent of the rest of the paper.

The next goal is the first proof that $\overline{\mathcal{Y}}_n$ is finite dimensional and is found in Section 7. In fact, we show that it has dimension $3n-4$, see Corollary 7.12. A key here is that the set of nongeometric actions in $\overline{\mathcal{Y}}_n$ behaves like a Z-set.

Sections 4-6 contain the technical results needed in Section 7. Of note are Corollary 5.2 and Theorem 6.2 which give splitting results for very small F_n-trees and dual band complexes.

Outer Space was introduced by Culler and Vogtmann [CV1]. They prove that $\mathcal{Y}_n$ is contractible, and use that to show that $Out(F_n)$ is of type VFL and $vcd(Out(F_n)) = 2n-3$. Culler and Morgan [CM] show that the projectivized space of small F_n-trees is compact and contains the space of free actions. Since Outer Space may be identified with projectivized free simplicial actions, $\overline{\mathcal{Y}}_n$ is compact. Steiner[St] and Skora [Sk1] show that $\overline{\mathcal{Y}}_n$ is contractible. The fixed set of a finite subgroup of $Out(F_n)$ acting on $\mathcal{Y}_n$ (respectively $\overline{\mathcal{Y}}_n$) is contractible. See White [W] and Krstić-Vogtmann [KV] (respectively White[W]). Rank two Outer Space is completely understood [CV2]. Cohen and Lustig [CL] show that $\overline{\mathcal{Y}}_n$ is contained in the projectivized space of very small actions and that a free simplicial action is in $\overline{\mathcal{Y}}_n$ if and only if it is very small. Every element of $Out(F_n)$ fixes a point of $\overline{\mathcal{Y}}_n$.

The starting point for this work is our previous paper [BF1].

1. Geometric Actions and Splittings of Trees and Band Complexes

The techniques of [BF1] are most easily used to analyze geometric actions. Also, many band complexes decompose into simpler pieces. In this section we make these remarks precise. Refer to [BF1] for definitions.

Definition 1.1. Let X be a band complex and T a $G = \pi_1(X)$-tree. A resolution $f : \tilde{X} \to T$ is *exact* provided for every G-tree T' and equivariant factorization

$$\tilde{X} \xrightarrow{f'} T' \xrightarrow{g} T$$

of f with f' a surjective resolution it follows that g is an isometry onto its image. In this case we say that T is *geometric*.

Remark 1.2. For any band complex X (or more generally any space equipped with a measured foliation[Sk2]), there is a natural pseudo-metric on the universal cover $\tilde{X}$ induced by integrating the measure along paths. The associated metric space T_X is a $\pi_1(X)$-tree called the *dual* of X. If X resolves some other tree T', then the natural map $\tilde{X} \to T_X$ is a resolution, and therefore an exact resolution

A decomposition of a band complex will engender a decomposition of the dual tree. Our decompositions will be of a rather simple nature that we explain now. Let T_i be G_i-trees for $i = 1, 2$, and let $J_i \subset T_i$ be isometric segments. Following Skora [Sk2], define the *free product* $T_1 *_{J_1=J_2} T_2$ to be the $G_1 * G_2$-tree obtained as follows. Let X_i be a *realization* of T_i, $i = 1, 2$, i.e. the space $\tilde{K}_i \times_{G_i} T_i$ for an Eilenberg-MacLane space K_i for G_i. There is an obvious equivariant map $\tilde{X}_i \to T_i$ from the universal cover of X_i to the tree T_i, and the point-preimages induce a measured foliation of $\tilde{X}_i$ and of X_i. The segment J_i can be identified with the projection of the segment $\{pt\} \times J_i$. Now define $T_1 *_{J_1=J_2} T_2$ to be the tree dual to the foliated space $X_1 \cup J \times [1, 2] \cup X_2$ where $J \times \{i\}$ is identified with J_i for $i = 1, 2$. Here, the band $J \times [1, 2]$ is foliated by segments $pt \times [1, 2]$.

Roughly, we can say that $T_1 *_{J_1=J_2} T_2$ arises from gluing T_1 and T_2 along segments J_1 and J_2. We may similarly define the result of gluing two segments J_1 and J_2 of the same tree T_0. This gives rise to the HNN-extension $T_0 *_{J_1=J_2}$. See Skora [Sk2] for details.

Definition 1.3. An F_n-tree T *splits* if either $T = T_1 *_{J_1=J_2} T_2$ or $T = T_0 *_{J_1=J_2}$ and the induced splitting of F_n is nontrivial.

Now we define the analogous notion for band complexes. If p is a path in a band complex X, let $l_X(p)$ denote its length with respect to the foliation.

Definition 1.4. Let X^i for $i = 1, 2$ be a band complex. Let A_i be a closed subinterval of a base of X^i. Suppose $A_1 = A_2$, i.e. there is given an isomorphism of the measured graphs A_1 and A_2. Suppose further that the image of A_i in $Dual(X^i)$ (first lift to $\tilde{X}^i$, then decompose) is an interval of length $l_{A_i}(A_i)$. Define the *free product* of X^1 and X^2 (over $A_1 = A_2$), denoted $X^1 *_{A_1=A_2} X^2$, to be the band complex $X^1 \cup_{A_1=A_2} X^2$. Similarly define an HNN-extension $X^1*_{A_1=A_2}$. We say that a band complex X *splits* if X is either a free product or an HNN-extension such that the induced splitting of $\pi_1(X)$ is nontrivial.

Proposition 1.5. *Suppose that the resolution $f : \tilde{X} \to T$ is exact. If X splits, then so does T.*

Proof. For the sake of notational simplicity, we assume that $X = X^1 *_{A_1=A_2} X^2$. Let $G_i = \pi_1(X^i)$ for $i = 1, 2$. Choose a lift $\tilde{A}_i$ of A_i to $\tilde{X}^i$. Let K_1 be the $K(G_1, 1)$ formed from X^1 by adding cells of dimension greater than 2. Let $T_1 = Dual(X^1)$. According to Remark 1.2, the natural map $f_1 : \tilde{X}^1 \to T_1$ is an exact resolution. Set $\Phi_1 : \tilde{X}^1 \to \tilde{K}_1 \times_{G_1} T_1$ by $\Phi_1(x_1) = [(x_1, f_1(x_1))]$. Similarly define Φ_2, etc. Notice that Φ_i induces an isometry of dual trees. According to Skora [Sk2], $T' = T_1 *_{f_1(\tilde{A}_1)=f_2(\tilde{A}_2)} T_2$ is dual to the free product K of $\tilde{K}_1 \times_{G_1} T_1$ and $\tilde{K}_2 \times_{G_2} T_2$ over images of $\tilde{A}_i$. By the universal properties of the free product T' naturally maps to T. Further, the maps Φ_i can be assembled to give a map from $\tilde{X}$ to $\tilde{K}$. Therefore, f factors as

$$\tilde{X} \to \tilde{K} \to T' \to T.$$

Since f is an exact resolution, T' and T are isometric. $\square$

Definition 1.6. We say that a tree T has a certain attribute if T admits an exact resolution from the universal cover of a band complex with that attribute. For example, "T has a thin component" means that T is dual to a resolving complex K with a thin component.

Definition 1.7. Let $f : \tilde{X} \to T$ be a resolution. Let b be a base of a band of X. Also, let $\overline{b}$ be the decomposition space (an interval) induced by restricting the foliation to b. First lifting b to $\tilde{X}$ and then applying f engenders a map from $\overline{b}$ to T. We say that *bases of X inject* (via f) if for each base, this map is an embedding.

By subdividing, we may assume that bases inject [BF1,Section 5]. The next proposition follows easily from the techniques of [BF1].

Proposition 1.8. *Suppose X resolves the small F_n-tree T. There are no axial components in the underlying union of bands Y. All bands of weight 0 in X that meet a minimal component of Y represent the trivial element of $\pi_1(X) = F_n$ and therefore can be collapsed. If all bases inject, then there are no weight $\frac{1}{2}$ bands that meet minimal components. If further nondegenerate arc stabilizers in T are primitive then X has no weight* $1/2$ *bands.*

Proof. The existence of an axial component forces the existence of a subgroup of F_n that maps onto a subgroup of $Isom(\mathbb{R})$ of rank greater than one and kernel an edge stabilizer (hence cyclic). Since F_n has no such subgroups, there are no axial components. Denote by z the element represented by a weight 0 band. Then z commutes with all elements of the form xzx^{-1}, where x is represented by a path in a leaf followed by an arc in the weight 0 base. In a free group this is possible only if z commutes with every such x. In the presence of minimality, the set of possible x's generates a nonabelian subgroup of $\pi_1(X)$ [BF1, Proposition 7.4], so that its centralizer is trivial.

If z is represented by a weight $\frac{1}{2}$ band, the above argument shows that z^2 is trivial, and hence so is z. But, then the base of the band cannot inject into T_X. A weight $1/2$ band represents a group element which leaves a nondegenerate arc invariant without fixing it, a contradiction. $\square$

Remark 1.9. Note that Proposition 1.8 implies, in particular, that if all components of Y are minimal, then the edge stabilizers in the dual tree T_X are trivial.

Definition 1.10. A band $B = b \times [-1, 1]$ of a band complex X is *naked* if the only cells of X that meet $b \times (-1, 1)$ are subdivision annuli [BF1, Section 5]. The band B is *very naked* if $b \times (-1, 1)$ meets no cell of X.

Remark 1.11. Under the hypothesis of Proposition 1.8, we may collapse the subdivision annuli along with the weight 0 bands and achieve that all naked bands that are contained in a thin component are very naked.

Proposition 1.12. *Suppose T is a small F_n-tree. If T has a thin component, then T splits.*

Proof. As usual , we may assume that all bases inject. In [BF1, Lemma 14.2] it is shown that applying the Rips machine to a thin component eventually creates a naked band. The proposition now follows from Remark 1.11 and Proposition 1.5. $\square$

2. Approximating Very Small Actions of Free Groups

In [CL] the notion of very small actions was introduced. An F_n-action is *very small* provided

- the edge stabilizers are either trivial or primitive cyclic, and
- $Fix(g)$ contains no triod when $g \neq 1$.

A simplicial F_n-action is very small if and only if in the quotient graph of groups, each edge group is either trivial or cyclic with generator primitive in both incident vertex groups, and whenever the groups of three distinct edges incident to a vertex are nontrivial, then they are not all conjugate in the vertex group. Cohen and Lustig go on to prove the following:

Theorem 2.1[CL].

(1) *A simplicial F_n-action can be approximated by a free action if and only if it is very small.*
(2) *If an F_n-action can be approximated by free actions, then it is very small.*

In this section we prove the following.

Theorem 2.2. *Every very small action of a free group F_n on an $\mathbb{R}$-tree T can be approximated by a free simplicial action. Equivalently, an F_n-action represents a point in $\overline{\mathcal{Y}}_n$ if and only if it is very small.*

Proof. In view of Theorem 2.1 of Cohen-Lustig, it suffices to approximate T by a very small simplicial action. Let $\mathcal{C}$ be a finite collection of elements of F_n and let $\epsilon > 0$. We will find a very small simplicial tree T' such that for all $\gamma \in \mathcal{C}$, $|l_T(\gamma) - l_{T'}(\gamma)| < \epsilon$.

If X is a band complex and $\gamma \in \pi_1(X)$, then let $l_X(\gamma) = inf\{l_X(p) | [p] = \gamma\}$. First we argue that T can be resolved by a finite band complex X so that the tree T_X dual to X satisfies $l_X(\gamma) = l_T(\gamma)$ for all $\gamma \in \mathcal{C}$. Indeed, first choose any resolution X. Then, for each $\gamma \in \mathcal{C}$, attach an annulus to X along a boundary curve with the attaching map given by γ, extending the lamination so that the free boundary component of the annulus has length $l_T(\gamma)$. We assume that all bases inject.

Now, we apply the Rips machine to X. By Proposition 1.8, no component of Y is of axial type. So we need to analyze the remaining three types. To fix ideas, we first present the "pure" cases.

Case 1. Each component of Y, the union of bands underlying X, is of thin type. This is the key case. We assume that only Process I is applied, and (by Proposition 1.8) that X has no bands of weight less than one. It follows that any band complex obtained from X by a collapse has no bands of weight less than one. Process I produces band complexes $X = X_0, X_1, X_2, \cdots$ with underlying union of bands $Y = Y_0, Y_1, Y_2, \cdots$. Since Y has no bands of weight less than 1, $Y_{i+1} \subset Y_i$. The limiting lamination L_∞ of Process I was introduced in [BF1, Lemma 14.1].

Sublemma 2.3. *There is an $m \geq 0$ and a very naked band B of X_m such that B meets L_∞ in at least one vertical arc.*

Proof. By [BF1, Proposition 11.2(2)], we may assume that no isolated (half-)bases appear in the sequence of X_i's. By [BF1, Proposition 11.2(4)], there is a band of X such that the number of components of the intersection of this band with X_i goes to infinity with i. Each of these components then has nonempty intersection with L_∞. $\square$

The proof shows in fact that there is a band that meets L_∞ in infinitely many vertical arcs, but we will not need this. Now we assume that X has a very naked band $B = [0, b] \times [-1, 1]$ as in Sublemma 2.3.

Let $x = min\{t \in [0, b] | \{t\} \times [-1, 1] \subset L_\infty\}$. Let B_i denote the very naked band of X_i that contains $\{x\} \times [-1, 1]$. The band B_i is a subset $[t_{-i}, t_i] \times [-1, 1]$ of B. If $t \in [t_{-i}, t_i)$, then let $X_{i,t}$ denote the band complex $X_i \setminus ([t_{-i}, t) \times (-1, 1))$. Notice that the composition of the inclusion of the universal cover of $X_{i,t}$ into the universal cover of X_i followed by a resolving map is again a resolving map. In the case $i = 0$, we may suppress the 0.

Sublemma 2.4. *Let $\eta > 0$. If $([0, s) \times [-1, 1]) \cap L_\infty = \emptyset$, then there is a number m and, for each $\gamma \in \mathcal{C}$, a representative loop p_γ in X_m such that $|l_{X_m}(\gamma) - l_{X_m}(p)| < \eta$ and p_γ misses $[0, s) \times (-1, 1)$. In particular, the duals of X and X_s are isometric.*

Proof. For each $\gamma \in \mathcal{C}$, choose a representative q_γ in X whose length is within η of $l_X(\gamma)$ and that meets $[0, s) \times (-1, 1)$ in finitely many vertical arcs. Let s' be the maximum $t \in [0, s)$ such that $\{t\} \times [-1, 1]$ is a of some segment of a q_γ. Since $([0, s) \times [-1, 1]) \cap L_\infty = \emptyset$, there is an m such that $Y_m \cap ([0, s'] \times (-1, 1)) = \emptyset$. Set p_γ to be the image of q_γ in X_m under the composition of the first m collapses. $\square$

Sublemma 2.5. *There is a $\delta > 0$ and a number m such that $l_X(\gamma) \leq l_{X_{m,x+\delta}}(\gamma) \leq l_X(\gamma) + \frac{\epsilon}{2}$ for all $\gamma \in \mathcal{C}$.* $\square$

Proof. Since X and X_m are related by Rips moves, they have the same length functions. Since $X_{m,x+\delta} \subset X_m$, the first inequality follows. For the second inequality, take the s in Sublemma 2.4 to be x and choose p_γ to be within $\frac{\epsilon}{4}$ of $l_X(\gamma)$. This produces the desired m. Let r be the number of vertical arcs of B_i that are segments of some p_γ. Then, in $l_{X_{m,x+\delta}}(p_\gamma) \leq l_{X_m}(p_\gamma) + 2r\delta$. So, choose $\delta = \frac{\epsilon}{4r}$. $\square$

Sublemma 2.6. *Let $\delta > 0$. After applying a sequence of Rips moves to $X_{x+\delta}$, we obtain a band complex that is either of the form $X^{-1} *_{x_{-1}=t_{-1}} [t_{-1}, t_1] *_{t_1=x_1} X^1$ or $X^1 *_{x_1=t_1} [t_1, t_2] *_{t_2=x_2}$.*

Proof. Let N be the first number such that the collapse from X_N to X_{N+1} involves $(x, x+\delta)\times[-1,1]$. Using the collapses in the sequence $X = X_0, X_1, X_2, \cdots$, we produce a sequence $X_{x+\epsilon} = X'_0, X'_1, X'_2, \cdots, X'_N$. The idea is to try to apply to $X_{x+\epsilon}$ the collapses performed on X.

The transition from X_i to X_{i+1} is a collapse in a long band. For convenience, we assume that this long band consists of just one band. (A collapse of a long band is a composition of collapses of bands, so there is no loss in this assumption.) We now inductively describe the transition from X'_i to X'_{i+1}. If the collapse from X_i to X_{i+1} does not involve $[0, x+\epsilon) \times (-1, 1)$, then the collapse may be applied to X'_i resulting in X'_{i+1}. If this collapse does involve $[0, x+\epsilon) \times (-1, 1)$, then set X'_{i+1}=X'_i. After applying the Nth collapse, there is a segment J of measured graph of X'_N whose intersection with bands is precisely its endpoints. Assume that $J \subset [0, b] \times \{1\}$ (the other case is handled symmetrically). The resulting complex is of the form $X^{-1} *_J X^1$ or $X^1 *_J$. Choose notation so that $(x, 1) \in X^1$. We need only check that the fundamental groups of X^i are nontrivial in the former case. Consider the leaf ℓ_∞ of L_∞ that contains $x \times [-1, 1]$. Let ℓ^1_∞ be the component of $\ell_\infty \cap X^1$ that contains $(x, 1)$. From [BF1, Lemma 14.1(3)], it follows that there is an arc in ℓ^1_∞ with endpoints two distinct points of a base of X. Indeed, if ℓ^1_∞ is not compact this statement is clear, otherwise the endpoints of ℓ^1_∞ are contained in $[x, x+\epsilon] \times \{-1, 1\}$. If ℓ^1_∞ has endpoints on both $[x, x+\epsilon] \times \{-1\}$ and $[x, x+\epsilon] \times \{1\}$, then the arc J is nonseparating, a contradiction. The loop in X^1 formed by connecting these endpoints in the base is nontrivial (since bases inject into T). The same argument works for X^{-1} (where $(x, 1)$ is replaced by $(x, -1)$). $\square$

We now finish case one. Choose m and δ as in Sublemma 2.5. The band complex $X_{m,x+\delta}$ is an approximation to X and after a finite number of moves splits as $X^1 *_J X^2$ or $X^1 *_J$. Each X^i requires fewer generators for its fundamental group than does X. Also, the characterization of leaves given in [BF1, Theorem 15.1] shows that the components of the union of bands Y^i underlying X^i are simplicial or thin (after splitting compact pushing saturated subsets of leaves). Since the inclusion $X_{m,x+\delta} \subset X_m$ induces a resolution, the arc stabilizers of $T_{X_{m,x+\delta}}$ are trivial. If $X_{m,x+\delta}$ has a thin component then we apply this procedure again starting with the complex $X_{m,x+\delta}$ instead of X and ϵ replaced by $\frac{\epsilon}{2}$. After iterating this procedure at most $n = rank(F_n)$ times, we obtain a simplicial complex whose dual approximates T. Since the edge stabilizers of this approximation are trivial, it is very small.

Case 2. X is of surface type. Then we can assume that X is a surface with boundary with a measured lamination whose leaves are disjoint from ∂X. It is well-known how to approximate the lamination by a measured closed 1-manifold (using train-tracks, and approximating the weights by rational numbers still satisfying the switch equations)[FLP]. The dual simplicial tree is very small.

Case 3. X is of simplicial type. The difficulty is that it might happen that the dual action is not very small. The dual tree T_X can be triangulated equivariantly so that the resolving map embeds each edge. We argue that there is a very small simplicial action that can be interpolated between T and T_X. We describe the argument in the language of the quotient graph $\Lambda = T_X/F_n$. First, by folding (cf. [BF2, Move 2]), we can arrange that all edge labels in Λ are either trivial or primitive cyclic subgroups of F_n. If T_X contains a triod fixed by a nontrivial group element, then the resolving map must identify the initial segments of two edges in the triod. Thus we can perform a fold in Λ, which may be partial (i.e. an edge is identified with a subinterval of another), that identifies edges with nontrivial labels. We need to argue that after a finite number of such folds, the action becomes very small.

Since Λ has only finitely many vertices, and their number decreases after a full fold, we may assume that all folds are partial. There is a natural 1-1 correspondence between the edges of graphs before and after a partial fold. Let N be the number of edges of Λ with a nontrivial label, and assume that we performed $N^2 + 1$ partial folds on Λ to obtain a new quotient graph Λ'. Then some edge e with a nontrivial label served as the 'short edge' in a partial fold at least $N+1$ times. The naturally induced map $\Lambda \to \Lambda'$ has the property that the preimage of an interior point of e contains at least $N+1$ points in the interiors of edges of Λ with nontrivial labels. Thus some edge of Λ with a nontrivial label maps to Λ' in such a way that two of its interior points are identified. We now reach a contradiction by observing that the label of e in Λ' is nonabelian. Indeed, there are distinct edges E and $g(E)$ in T_X for some $g \in F_n$ such that both $Stab(E)$ and $Stab(g(E)) = gStab(E)g^{-1}$ stabilize e. Since $Stab(E)$ is primitive cyclic, the group generated by $Stab(E)$ and $Stab(g(E))$ is nonabelian.

Each fold can be realized in X by attaching a band and a 2-cell killing the extra generator.

Case 4. The general case. Each component of Y is of surface, thin, or simplicial type. On each thin component perform the operation as discussed in case 1.

At this stage, X can be viewed as $S \cup R$, where R is of simplicial type and where S is a surface (possibly disconnected), the fundamental group of each component of S injects in $\pi_1(X)$,and the lamination restricts to a geodesic-like lamination on S. Furthermore, we can assume that the boundary of S is bicollared in X (if necessary, include a collar of the boundary of S in R).

Next, work on the simplicial components as in case 3, until they all become very small. Finally, approximate simplicially all surface components making sure that no leaf is parallel to the boundary. We leave it to the reader to check that the result is very small. $\square$

3. Geometric vs. Nongeometric Free Actions of Free Groups

In this section we produce examples of nongeometric actions of the free group in the boundary of Outer Space. Furthermore, we give an algorithm for deciding whether or not the attracting fixed point of a given exponentially growing irreducible outer automorphism is geometric. The rest of this paper is independent of this section.

We use some terminology of [BH1]. Let $f : \Lambda \to \Lambda$ be a stable train-track map defined on a graph Λ representing an exponentially growing irreducible outer automorphism $\mathcal{O}$ of the free group $\pi_1(\Lambda)$. The existence of f is shown in [BH1, Proposition 3.3]. Immersed paths $\alpha_1, \alpha_2, \ldots, \alpha_k$ in Λ form an *orbit of periodic Nielsen paths* if $f(\alpha_i)$ is homotopic rel endpoints to $\alpha_{i+1 mod k}$. This orbit is *indivisible* if α_1 is not a concatenation of subpaths that belong to orbits of periodic Nielsen paths. One can argue as in [BH1, Lemma 3.4] that each path in an indivisible orbit of Nielsen paths has exactly one illegal turn.

Lemma 3.9 of [BH1] shows that Λ supports at most one indivisible (fixed) Nielsen path. The same argument can be used to show that Λ supports at most one indivisible orbit of periodic Nielsen paths (up to cyclic reordering and change of orientation).

Using the techniques of [BH1] it is not hard to prove the following.

Proposition 3.1.

(1) *If Λ contains no indivisible orbits of periodic Nielsen paths, then for any loop [path] β in Λ there exists $n > 0$ such that $f^n(\beta)$ is homotopic [rel endpoints] to a legal loop [path].*
(2) *If Λ contains a unique indivisible orbit $\alpha_1, \ldots, \alpha_k$ of periodic Nielsen paths, then for any loop [path] β in Λ there exists $n > 0$ such that $f^n(\beta)$ is homotopic [rel endpoints] to an immersed loop [path] which is a concatenation of legal paths and α_i's with illegal turns occurring only within the α_i's.*

Recall that $\mathcal{O}$ acts on Outer Space by change of marking. The sequence $\{\mathcal{O}^n(\Lambda)\}_{n=1}^{\infty}$ converges in compactified Outer Space to a point in the boundary represented by an action of $\pi_1(\Lambda) = F_n$ on an $\mathbb{R}$-tree $T_{\mathcal{O}}$. Here, Λ is viewed as a metric graph where the lengths of edges are chosen so that f expands each edge uniformly by $\lambda > 1$. The translation length in T of a conjugacy class α is obtained as the limit of the monotonically nonincreasing sequence $\{length(f^n(\alpha))/\lambda^n\}$, where $length(\beta)$ denotes the length in Λ of the curve with no backtracking representing the conjugacy class β. For legal loops this sequence is constant. The limiting $\mathbb{R}$-tree $T_{\mathcal{O}}$ does not depend on the choice of representative $f : \Lambda \to \Lambda$; indeed, for every H in outer space $\mathcal{O}^n(H) \to T_{\mathcal{O}}$ as $n \to \infty$. Furthermore, for any conjugacy class β we have $length(\mathcal{O}(\beta)) = \lambda\ length(\beta)$.

The purpose of this section is to study when $T_{\mathcal{O}}$ is geometric.

Theorem 3.2. *$T_{\mathcal{O}}$ is geometric if and only if Λ contains an indivisible orbit of periodic Nielsen paths.*

Example 3.3. The automorphism of F_3 given by $a \to b$, $b \to c$, and $c \to ab$ is represented on the bouquet of 3 circles labeled a, b, and c by the map given by the above formula.

This map is a stable train-track map (the only non-degenerate illegal turn is $\{\overline{a}, \overline{c}\}$) with no periodic Nielsen paths.

Example 3.4. The obvious representation $f : \Lambda \to \Lambda$ of the automorphism $a \to ac$, $b \to a$, and $c \to b$ on the bouquet of 3 circles is a stable train-track map. The illegal turns are formed by pairs of edges in $\{a, b, c\}$. The fourth power of f maps a to $acbaac$ and thus has two fixed points in the interior of a. Subdivide at the first fixed point, so that $a = a_1 a_2$ where $f^4(a_1) = acba_1$ and $f^4(a_2) = a_2 ac$. Similarly, we can write $c = c_1 c_2$ where $f^4(c_1) = ac_1$ and $f(c_2) = c_2 b$. Now $\overline{a}_1 \overline{b} a c_1$ is homotopically fixed under f^4. Therefore, this path together with its 3 iterates under f forms an indivisible orbit of periodic Nielsen paths of period 4. This is the only such orbit. No essential loop is a concatenation of these 4 paths, and therefore the limiting group action on $T_{\mathcal{O}}$ is free.

Remark 3.5. If Λ contains a unique indivisible orbit $\alpha_1, \dots, \alpha_k$ of periodic Nielsen paths whose concatenation is a loop fixed by f (up to homotopy), then $\mathcal{O}$ can be realized as a pseudo-Anosov homeomorphism on the surface with one boundary component obtained from Λ by attaching an annulus along this loop [BH1,Proposition 4.5]. More generally, if the α_i's concatenate to give more than one loop, then $\mathcal{O}$ can be realized as a homeomorphism of a surface with more than one boundary component. In this case $T_{\mathcal{O}}$ is dual to the unstable lamination of the homeomorphism.

One can construct a sequence of finer and finer simplicial approximations to $T_{\mathcal{O}}$ as follows. Let $\Lambda_0 = \Lambda$ and, in general for $m > 0$, $\Lambda_m = \mathcal{O}^m(\Lambda)/\lambda^m$. Then f followed by scaling down by λ induces morphisms

$$\tilde{\Lambda}_0 \xrightarrow{\tilde{f}_1} \tilde{\Lambda}_1 \xrightarrow{\tilde{f}_2} \tilde{\Lambda}_2 \to \dots .$$

The tree $T_{\mathcal{O}}$ can be described as the Gromov limit [Pa] of this sequence. Notice that every legal path in $\tilde{\Lambda}_m$ is mapped isometrically, and hence there is a limiting map $\Psi_m : \tilde{\Lambda}_m \to T_{\mathcal{O}}$.

The proof of one half of Theorem 3.2 follows easily from the next proposition.

Proposition 3.6. *Suppose that Λ contains no indivisible orbits of periodic Nielsen paths, and that a finite band complex K resolves $T_{\mathcal{O}}$. Then the resolution map factors through $\tilde{\Lambda}_m$ for a sufficiently large m. In particular, $T_{\mathcal{O}}$ is not geometric.*

Proof. Let $\Xi : \tilde{K} \to T_{\mathcal{O}}$ be a resolution. For notational simplicity, we may assume that Ξ embeds components of the lifted measured graph $\tilde{\Gamma}$, which are arcs. For every vertex $v \in \tilde{K}$ (i.e. a vertex of $\tilde{\Gamma}$, corner of a band, or a 0-cell of $\tilde{K}$) choose a point $\Phi_0(v) \in \tilde{\Lambda}_0$ so that Φ_0 is equivariant, and so that $\Psi_0 \Phi_0 = \Xi$ on the vertices. Now find $m > 0$ so that for every edge e in $\tilde{K}$ (i.e. a component of $\tilde{\Gamma}$, a vertical boundary component of a band, or a 1-cell in $\tilde{K}$) the arc in $\tilde{\Lambda}_m$ joining the two points in $\tilde{f}_m \dots \tilde{f}_2 \tilde{f}_1 \Phi_0(\partial e)$ is legal (or constant). It is now straightforward to extend $\tilde{f}_m \dots \tilde{f}_2 \tilde{f}_1 \Phi_0$ to an equivariant map $\Phi_m : \tilde{K} \to \tilde{\Lambda}_m$, thus yielding a resolution of Λ_m with $\Psi_m \Phi_m = \Xi$. $\square$

Proof of Theorem 3.2. It remains to consider the case when Λ contains an indivisible orbit $\alpha_1, \dots, \alpha_k$ of periodic Nielsen paths. To each of the k paths attach a disk along an arc in the boundary. The disk is foliated so that the leaves join points on the path equidistant from the illegal turn. Thus obtained foliated complex K can be given the structure of a band complex. It resolves $T_{\mathcal{O}}$, and the loops that are concatenations of Nielsen paths and legal arcs as above have the same length in $T_{\mathcal{O}}$ as in K. Furthermore, there is a map $\hat{f} : K \to K$ sending leaves to leaves and extending f. Now, $f : \Lambda \to \Lambda$ can be represented as the composition of maps each of which folds an illegal turn, followed by the uniform scaling of the metric by factor λ. As in [BH1, Lemma 3.9] one easily argues that every one of the folds occurs at the illegal turn of a path in the indivisible orbit of Nielsen paths. This induces a similar factorization of $\hat{f} : K \to K$, which implies that $l_K(\hat{f}(\beta)) = \lambda\, l_K(\beta)$ holds in K (as well as $l_{T_{\mathcal{O}}}(\mathcal{O}(\beta)) = \lambda\, l_{T_{\mathcal{O}}}(\beta)$). (For the definition of l_K see Section 2.) Thus, the lengths in K and in $T_{\mathcal{O}}$ agree. □

4. A free group decomposition lemma

In this head we prove a technical lemma needed in the sequel.

Lemma 4.1. *Suppose that Y is a finite (possibly disconnected) graph, and S a compact (possibly disconnected) surface. Let $f : \partial S \to Y$ be a map that is essential on each boundary component. Assume that the adjunction space $X = S \cup_f Y$ has free fundamental group. Then there is a homotopy equivalence $\psi : Y \to S^1 \vee Y'$ to the wedge of the circle and a graph so that the composition $\psi f : \partial S \to S^1 \vee Y'$ is homotopic to a map that sends one boundary component homeomorphically onto S^1, and sends all other boundary components into Y'.*

Example 4.2. Suppose a free group is represented as $A *_{\mathbb{Z}} B$ for finitely generated free groups A and B, and suppose the generator for $\mathbb{Z}$ corresponds to $a \in A$ and $b \in B$. Then either a is a basis element of A or b is a basis element of B.

To see this, start with the disjoint union Y of two finite graphs representing A and B, and then attach the annulus along its boundary via a and b. Then apply the lemma.

Proof. Let $h : X \to G$ be a map to a graph that induces an isomorphism between fundamental groups. Replacing Y by a homotopy equivalent graph if necessary, we may assume that h restricted to Y is an immersion, and that f is an immersion. It suffices to argue that at least one edge in Y is crossed geometrically exactly once by f. Suppose not. Let Z be a finite subset of Y that intersects each edge in an interior point. We may assume that h is transverse to Z. The assumption that no edge of Y is crossed exactly once by f guarantees that the graph $h^{-1}(Z)$ has no valence 1 vertices. In particular, there is a loop $\gamma \subset h^{-1}(Z)$ that is not a trivial loop in the interior of S. We will argue that γ is essential, contradicting the assumption that h is π_1-injective.

First, notice that if $\gamma_0 \subset S$ is an arc of γ whose endpoints are in ∂S, then γ_0 together with an arc in ∂S could not bound a disk in S. For otherwise, find an innermost arc in $h^{-1}(Z)$ that together with an arc in ∂S bounds a disk in S, and conclude that the composition $hf : \partial S \to G$ is not an immersion, contrary to the hypotheses.

Next, notice that if $\gamma_0 \subset S$ is an arc of γ whose endpoints are in S, and which switches sheets at a point of Y, then γ_0 is not homotopic rel endpoints to an arc contained in the interior of S (since f is an immersion).

The above two facts imply that the lift of γ to the universal cover $\tilde{X}$ of X never returns to the component of the preimage of S it leaves. In particular, γ is essential. $\square$

5. The structure of band complexes

Definition. A measured lamination on a surface is *geodesic-like* if it admits a hyperbolic metric such that the lamination is geodesic and filling (i.e., the complement does not contain an essential nonperipheral loop).

Recall [Ha] that if X is a band complex that resolves an F_n-tree T, and if S is a laminated surface component of the union of bands with inessential boundary components capped off, then the lamination on S is geodesic-like (this uses only the fact that F_n is torsion-free; there is a straightforward generalization to groups with torsion using orbifolds).

Proposition 5.1(Structure). *Let X be a band complex with only surface and simplicial components and whose universal cover resolves a tree whose arc stabilizers are primitive cyclic or trivial. Then there is another band complex with the same dual as X of the form $(S \cup A \cup \Gamma) \cup_f G$ such that*

1. *S is a compact surface with a geodesic-like lamination,*
2. *Γ is a finite real graph,*
3. *G is a finite graph with no valence 1 vertices and empty lamination,*
4. *A is a finite disjoint union of annuli laminated by essential loops, and*
5. *$f : \partial S \cup \partial A \cup F \to G$ where F is a finite subset of $S \cup A \cup \Gamma$ and f is essential on each component of $\partial S \cup \partial A$.*

Proof. By applying the Rips machine we may arrange that the union of bands in each simplicial component is a real graph with weight 0 bands (i.e., annuli) attached (recall that by Proposition 1.8 there are no weight 1/2 bands). Subdivide each weight 0 band so that if two bases overlap, they coincide. Then slide bands with coinciding bases over each other so that exactly one band in each family with coinciding bases generates the stabilizer of the edge of the tree corresponding to the base, and the others are trivial. Collapse the trivial ones. We have now achieved that the annuli have disjoint interiors. Take A to be the union of these annuli.

We may also arrange that the union of bands in each surface component is a laminated surface. Take S to be the union of these surfaces and take Γ to be the union of the closures of the complements of the annuli in the real graphs above. Replace each component C of the closure of the complement of $S \cup A \cup \Gamma$ in X by a bouquet of $r(C)$ circles where $r(C)$ is the rank of the image of $\pi_1(C)$ in $\pi_1(X)$ (A bouquet of 0 circles is a point). Take G to be the union of these bouquets of circles. Our band complex now has the desired form except perhaps for the conditions on f. By enlarging G, we may arrange that f is defined on all of ∂S. By capping off those boundary components of S that are inessential in X, we achieve the desired form. $\square$

A *generalized band* B is (finite tree)$\times I$ laminated by $pt \times I$. A band complex X *contains a very naked generalized band* $B = \Gamma \times I$ if it is of the form $X = Y \cup B$ and $Y \cap B = \Gamma \times \partial I$.

Corollary 5.2. *Let X be a band complex whose universal cover resolves a tree whose arc stabilizers are primitive cyclic or trivial. Then X is equivalent to a band complex that has one of the following forms.*

(1) *X contains a point that induces a nontrivial splitting.*
(2) *X contains a very naked generalized band giving rise to a nontrivial splitting.*

Proof. If X has a thin component, see [BF1]. So suppose it does not. We may assume that X is as in the proposition. Further take F to be of minimal cardinality. In this case, if F is not empty, then a point in F determines a nontrivial splitting.

If F is empty then X has the form as in Lemma 4.1. There are now three cases. If there is an edge of G that is not in the image of f, then the action clearly splits. If the distinguished circle C provided in Lemma 4.1 is not a component of G, then a valence greater than 2 vertex of G that is on C provides a splitting. The final possibility is that C is a component of G. In this case C corresponds to a boundary component of $S \cup A$. If $C \subset S$, an essential arc with endpoints on C provides a splitting. If $C \subset A$, collapse the annulus containing C, and use induction on the number of components of A. □

6. Geometric actions split

If T is an F_n-tree and $x \in T$, we can form another F_n-tree $T' = T *_{x=\partial_- K} K$, where we may view the closed interval K as a tree equipped with the action of the trivial group or, if $Stab(x) = \mathbb{Z}$, as a trivial $\mathbb{Z}$-tree. The induced (trivial) splitting of F_n is $F_n * < 1 >$ in the former case and $F_n *_\mathbb{Z} *\mathbb{Z}$ in the latter. We call this operation *adding a stick*. A tree is *nearly minimal* if it can be obtained from a minimal tree by finitely many such operations.

Proposition 6.1. *Suppose X is a band complex whose universal cover resolves a tree with arc stabilizers primitive cyclic or trivial. Then*

(1) *for any two points in the universal cover of X there is a path joining them so that the integral of the transverse measure along the path realizes the infimum of such integrals over all paths joining the two points and*
(2) *$Dual(X)$ is nearly minimal.*

Proof. Note that if X is simplicial, the conclusions are obvious. In particular, if $n = 0$, by local injectivity X is simplicial. Now induct on n. By the above corollary, there is either a point or a very naked generalized band giving rise to a splitting. The hypotheses on X are inherited by the pieces. By induction on n, the pieces satisfy the conclusion of the proposition. A routine exercise establishes the conclusions for X. □

This section is devoted to showing that all geometric actions split. See Definition 1.3. In fact, the pieces into which they split are nearly minimal, so that an inductive scheme can be used to prove things about geometric actions.

Theorem 6.2. *Let X be dual to very small minimal tree T. Then X splits. Furthermore, T has a splitting of one of the following forms with R and R' very small geometric minimal trees or points, $r, \hat{r} \in R$, $r' \in R'$, J and J' closed intervals, K a closed interval viewed as* *a tree equipped with the action of the trivial group, and L and L' closed intervals viewed as trivial $\mathbb{Z}$-trees.*

(1) $R *_{J=J'} R'$. *The corresponding splitting of F_n is $F_k * F_{n-k}$.*
(2) $R*_{J=J'}$. *The corresponding splitting of F_n is $F_{n-1}*_{<1>}$.*
(3) $(R *_{r=\partial_- K} K)*_{J=J'}$ *where $J = K * \hat{J} * K'$ with K' a translate of K in $R *_{r=\partial_- K} K$ and $\hat{J}$ a subinterval of J contained in R. The corresponding splitting of F_n is $(F_{n-1}* < 1 >)*_{<1>}$.*
(4) $(R *_{r=\partial_- K} K)*_{J=J'}$ *where $J = K * \hat{J}$, $J' = \hat{J}' * K'$ with K' a translate of K in $R *_{r=\partial_- K} K$, the gluing done so as to identify the endpoints of J on $\hat{J}$ and K respectively with the endpoints of J' on K' and $\hat{J}'$, and length$(J) \geq 2$ length(K). The corresponding splitting of F_n is $(F_{n-1}* < 1 >)*_{<1>}$.*
(5) $R *_{r=\partial_- K} K *_{\partial_+ K=r'} R'$. *The corresponding splitting of F_n is $F_k* < 1 > *F_{n-k}$.*
(6) $R *_{r=\partial_- L} L *_{\partial_+ L=\partial_- K} K *_{\partial_+ K=r'} R'$ *where r is an endpoint of an interval stabilized by $\mathbb{Z}$. The corresponding splitting of F_n is $F_k *_{\mathbb{Z}} \mathbb{Z}* < 1 > *F_{n-k}$.*
(7) $R *_{r=\partial_- L} L *_{\partial_+ L=\partial_- K} K *_{\partial_+ K=\partial_- L'} L *_{\partial_+ L'=r'} R'$ *where r and r' are endpoints of intervals stabilized by $\mathbb{Z}$. The corresponding splitting of F_n is $F_k *_{\mathbb{Z}} \mathbb{Z}* < 1 > *\mathbb{Z} *_{\mathbb{Z}} F_{n-k}$.*
(8) $(R *_{r=\partial_- K} K)*_{\partial_+ K=\hat{r}}$. *The corresponding splitting of F_n is $(F_{n-1}* < 1 >)*_{<1>}$.*
(9) $(R *_{r=\partial_- L} L *_{\partial_+ L=\partial_- K} K)*_{\partial_+ K=\hat{r}}$ *where r is an endpoint of an interval in R stabilized by $\mathbb{Z}$. The corresponding splitting of F_n is $(F_{n-1} *_{\mathbb{Z}} \mathbb{Z}* < 1 >)*_{<1>}$.*
(10) $(R *_{r=\partial_- L} L *_{\partial_+ L=\partial_- K} K *_{\partial_+ K=\partial_- L'} L')*_{\partial_+ L'=\hat{r}}$ *where r and $\hat{r}$ are endpoints of intervals stabilized by $\mathbb{Z}$. The corresponding splitting of F_n is $(F_{n-1} *_{\mathbb{Z}} \mathbb{Z}* < 1 > *\mathbb{Z})*_{\mathbb{Z}}$.*

Proof. By Corollary 5.2, either X has a very naked band or a point that induces a splitting. First suppose that X has a very naked band. In the case that this band is separating, $T = T_1 *_{J_1=J_2} T_2$ where each T_i are nearly minimal. Let $G_1 * G_2$ be the corresponding splitting for F_n. Since T is minimal, T_i is the convex hull of J_i and a minimal G_i-tree. In fact, T has a similar splitting where either J_i is degenerate or J_i does not meet a valence 1 point of T_i. Indeed, if say $T_1 = T_1' * K$ and $J_1 = J_1' * K$, then $T = T_1' *_{J_1'=J_2'} T_2'$. The tree T is then of the form as in item (1) or (5). If the band is not separating then $T = T_1*_{J_1=J_2}$. As above, if some J_i meets a valence 1 point of T_1 then we may find subintervals that also give rise to the splitting except in two cases. One case, corresponding to item (3) above, is where both endpoints of one of the J_i's say J_1 meets a valence 1 point of T_1. The other case, corresponding to item (4), is where an endpoint of J_1 and also one of J_2 meet translates of the same valence 1 point of T_1. Thus, $T = (T_1 * K)*_{J_1=J_2}$, $J_1 = K * J_1'$, and $J_2 = J_2' * K'$ where K' is a translate of K. If the gluing is done so that the segments of J_1 and J_2 corresponding to K are identified, then T is not minimal. Further, if the length of J_i is less than twice the length of K, then the midpoint of J_i is fixed by an element fixing only the midpoint, but whose square fixes a nondegenerate interval about the midpoint. This cannot occur in a very small tree. The case where no J_i meets an valence 1 point of T_1 covered by item (2). The case where the splitting guaranteed by Corollary 5.2 is over a point is easier and left to the reader. $\square$

7. The Dimension of the Compactified Outer Space

We assume the reader is familiar with the basics of $\mathbb{R}$-trees [CM], band complexes [BF1], as well as rank n outer space $\mathcal{Y}_n$ [CV1], and its closure $\overline{\mathcal{Y}}_n$ [CM]. This section is devoted to showing that the dimension of $\overline{\mathcal{Y}}_n$ is $3n-4$. In Section 2 we showed that $\overline{\mathcal{Y}}_n$ is the projectivization of the space $\overline{\mathcal{X}}_n$ of very small actions. For convenience we work with $\overline{\mathcal{X}}_n$. We will also use the space $\mathcal{A}_n$ of all F_n-trees and the space $\mathcal{X}_n$ of free simplicial actions. All spaces are separable and metrizable.

Define the universal bundle $\tilde{\mathcal{X}}_n$ over $\overline{\mathcal{X}}_n$ to be $\{(T,x)|T \in \overline{\mathcal{X}}_n, x \in T\}$, with bundle map $\pi_n : \tilde{\mathcal{X}}_n \to \overline{\mathcal{X}}_n$. See White [W] for the topology and details.

Lemma 7.1. *The space $\tilde{\mathcal{X}}_n$ is σ-compact.*

Proof. For $n = 1$, $\tilde{\mathcal{X}}_n$ is homeomorphic to $\mathbb{R}^2$. For $n > 1$, consider pairs of noncommuting elements α and β in F_n. Let $U = U_{\alpha,\beta}$ be the open subset of $\overline{\mathcal{X}}_n$ where these elements are hyperbolic. The set $\tilde{U} = \{(T,x)|T \in U, x \in Axis(\alpha)\}$ is a trivial line bundle over U. Indeed, a section is constructed by considering the midpoint of the set of points on $Axis(\alpha)$ that are closest to $Axis(\beta)$. Thus, $\tilde{U}$ is σ-compact. The space $\tilde{\mathcal{X}}_n$ is a countable union of such line bundles. $\square$

The rest of this section is devoted to the proof of the following theorem.

Theorem 7.2. $dim(\overline{\mathcal{X}}_n) = 3n-3$.

Since $\overline{\mathcal{X}}_n$ contains $\mathcal{X}_n$, it is clear that $dim\ \overline{\mathcal{X}}_n \geq 3n-3$. The case $n = 2$ is contained in [CV2]. For the remainder of this section, we will assume that $dim(\overline{\mathcal{X}}_k) = 3k-3$ for $k < n$.

Lemma 7.3. $dim(\tilde{\mathcal{X}}_n) \leq dim(\overline{\mathcal{X}}_n) + 1$.

Proof. The bundle projection $\tilde{\mathcal{X}}_n \to \overline{\mathcal{X}}_n$ is a map between σ-compact spaces with 1-dimensional fiber. The lemma follows by [HW, Theorem VI 7]. $\square$

Next we define spaces that parametrize splittable actions. Once and for all we fix a basis $\{x_1, \ldots, x_n\}$ for F_n. We use d_k to denote the metric in the appropriate fiber of $\tilde{\mathcal{X}}_k$. We also abuse notation slightly and identify a point in the universal bundle with the point in the tree it determines. There are 10 parameter spaces, each corresponding to a case in Theorem 6.2. We work out cases (1), (2), (5), and (8) in detail, and leave the others to the reader.

For $1 \leq k \leq n-1$ and $l = n-k$, define

$$P_{k,l} = \{(a_1, a_2, b_1, b_2) \in \tilde{\mathcal{X}}_k \times \tilde{\mathcal{X}}_k \times \tilde{\mathcal{X}}_l \times \tilde{\mathcal{X}}_l | \pi_k(a_1) = \pi_k(a_2),$$

$$\pi_l(b_1) = \pi_l(b_2), d_k(a_1,a_2) = d_l(b_1,b_2),$$

$$if\ k = 1,\ then\ a_1 = 0,\ if\ l = 1,\ then\ b_1 = 0\}/(a_1,a_2,b_1,b_2)\ (a_2,a_1,b_2,b_1),$$

$$Q_{k,l} = \{(a, b, t) \in \tilde{\mathcal{X}}_k \times \tilde{\mathcal{X}}_l \times \mathbb{R}_{\geq 0}\},$$

$$P_{n-1} = \{(a_1, a_2, b_1, b_2) \in \tilde{\mathcal{X}}_{n-1} \times \tilde{\mathcal{X}}_{n-1} \times \tilde{\mathcal{X}}_{n-1} \times \tilde{\mathcal{X}}_{n-1} | \pi_k(a_1) = \pi_k(a_2) = \pi_l(b_1) = \pi_l(b_2),$$

$$d_k(a_1, a_2) = d_l(b_1, b_2),\ if\ n = 2,\ then\ a_1 = 0\}/(a_1, a_2, b_1, b_2)\ (a_2, a_1, b_2, b_1),$$

and

$$Q_{n-1} = \{(a, b, t) \in \tilde{\mathcal{X}}_{n-1} \times \tilde{\mathcal{X}}_{n-1} \times \mathbb{R}_{\geq 0} | \pi_1(a) = \pi_1(b)\}.$$

Lemma 7.4. *For $k, l < n$, $P_{k,l}, Q_{k,l}, P_{n-1}, Q_{n-1}$ are σ-compact and $dim(P_{k,l}) \leq 3(k + l) - 3$, $dim(Q_{k,l}) \leq 3(k + l) - 3$, $dim(P_{n-1}) \leq 3n - 3$, $dim(Q_{n-1}) \leq 3n - 3$.*

Proof.

$$dim(P_{k,l}) \leq dim(\tilde{\mathcal{X}}_k) + dim(\tilde{\mathcal{X}}_l) + 1,$$

$$dim(Q_{k,l}) \leq dim(\tilde{\mathcal{X}}_k) + dim(\tilde{\mathcal{X}}_l) + 1,$$

$$dim(P_{n-1}) \leq dim(\tilde{\mathcal{X}}_{n-1}) + 2,$$

$$dim(Q_{n-1}) \leq dim(\tilde{\mathcal{X}}_{n-1}) + 2.$$

□

For $k + l = n$, $k, l > 0$ define $\phi_{k,l} : P_{k,l} \to \mathcal{A}_n$ by $\phi_{k,l}(a_1, a_2, b_1, b_2) = T_a *_{[a_1,a_2]=[b_1,b_2]} T_b$ where $T_a = \pi_k(a_1) = \pi_k(a_2)$ and $T_b = \pi_l(b_1) = \pi_l(b_2)$. The group F_n is identified with $F_k * F_l$ via $F_k = \langle x_1, \ldots, x_k\rangle$ and $F_l = \langle x_{k+1}, \ldots, x_n\rangle$.

Similarly, let $\psi_{k,l} : Q_{k,l} \to \mathcal{A}_n$ be defined by $\psi_{k,l}(a, b, t) = T_a *_{a=0} [0, t] *_{t=b} T_b$, $\phi_{n-1} : P_{n-1} \to \mathcal{A}_n$ by $\phi_{n-1}(a_1, a_2, b_1, b_2) = T *_{[a_1,a_2]=[b_1,b_2]}$ and $\psi_{n-1} \to \mathcal{A}_n$ by $\psi_{n-1}(a, b, t) = T *_{a=0} [0, t] *_{t=b}$.

Lemma 7.5. *$\phi_{k,l}$, $\psi_{k,l}$, ϕ_{n-1}, and ψ_{n-1} are 1-1 and continuous.*

Proof. Continuity is straightforward. We will show that $\phi_{k,l}$ is 1-1, the others being similar. We need a sublemma whose proof is left to the reader. If d is a *direction* in T, i.e. a germ of arcs based at a point, then let T_d denote the set of all points in T to which d points.

Sublemma 7.6. *Suppose T is a minimal, hyperbolic G-tree. Let d be a direction in T. Then, T_d contains an axis of a hyperbolic element.*

Let (a_1, a_2, b_1, b_2) and (a_1', a_2', b_1', b_2') be two distinct elements in $Domain(\phi_{k,l})$. It is clear that $\pi_k(a_1) = \pi_k(a_2) = \pi_k(a_1') = \pi_k(a_2')$ and $\pi_l(b_1) = \pi_l(b_2) = \pi_l(b_1') = \pi_l(b_2')$. Call these trees T_a and T_b respectively. We may assume that $d_k(a_1, a_2) \geq d_k(a_1', a_2')$, and so $d_k(b_1, b_2) \geq d_k(b_1', b_2')$. Let d_a be a direction in T_a and d_b be a direction in T_b such that

- d_a is based at a_1 or a_2, d_b is based at b_1 or b_2,

• d_a points away from the union $[a_1, a_2] \cup [a'_1, a'_2]$, d_b points away from the union $[b_1, b_2] \cup [b'_1, b'_2]$

• either $d_a(d, [a'_1, a'_2]) > 0$, or $d_b(d, [b'_1, b'_2]) > 0$.

Choose $\alpha \in F_k$ with axis in T_{d_a} and $\beta \in F_l$ with axis in T_{d_b}. Now, $l_{\phi_{k,l}(a_1,a_2,b_1,b_2)}(\alpha\beta) < l_{\phi_{k,l}(a'_1,a'_2,b'_1,b'_2)}(\alpha\beta)$. $\square$

Corollary 7.7. *The subset S of splittable, very small F_n-actions corresponding to cases (1), (2), (5), and (8) is σ-compact and has dimension $3n-3$.*

Proof. The images of $\phi_{k,l}$, $\psi_{k,l}$ $(k+l=n)$, ϕ_{n-1}, and ψ_{n-1} may not be contained in $\overline{\mathcal{X}}_n$, but their intersections with $\overline{\mathcal{X}}_n$ are σ-compact. (The class of σ-compact spaces is closed under passing to images and closed subsets.) Furthermore, S is covered by the countably many $Aut(F_n)$-translates of these images, hence S is σ-compact. (Countable union of σ-compact spaces is σ-compact.) The corollary now follows from [HW, Theorem III.2]. $\square$

Theorem 7.8. *The subspace of $\overline{\mathcal{X}}_n$ consisting of trees that admit a splitting as in Theorem 6.2 is σ-compact, has dimension $3n-3$, and contains all geometric actions in $\overline{\mathcal{X}}_n$.*

Proof. The other cases are treated similarly, and the details are left to the reader. $\square$

Proposition 7.9. *Given elements $\gamma_1, \cdots, \gamma_m \in F_n$, there is a map $\Phi : \mathcal{A}_n \to \mathcal{A}_n$ such that*

• *$\Phi(T)$ is geometric for all $T \in \mathcal{A}_n$,*

• *$l_T(\gamma_i) = l_{\Phi(T)}(\gamma_i)$, and*

• *$\Phi(\mathcal{X}_n) \subset \mathcal{X}_n$, and so $\Phi(\overline{\mathcal{X}}_n) \subset \overline{\mathcal{X}}_n$.*

Proof. Step 1: Fix a marked rose of rank n. Allowing the lengths of edges to vary describes a subset Δ of $\overline{\mathcal{X}}_n$ that is a copy of the orthant of $\mathbb{R}^n$ minus the origin. For every $T \in \mathcal{X}_n$ there is a point $\sigma(T)$ in Δ and an equivariant map $\Sigma(T) : \sigma(T) \to T$ which sends edges of $\sigma(T)$ isometrically. Furthermore, $\Sigma(T)$ varies continuously with T in the space of equivariant maps from $\overline{\mathcal{X}}_n$ to $\overline{\mathcal{X}}_n$. See Skora [Sk1].

Step 2: The next goal is to correct the lengths of the γ_i's by attaching foliated disks to the rose $\sigma(T)$ in a manner that is continuous in T.

First assume that the collection $\{\gamma_i\}$ of the proposition consists of a single element γ with $l_{\sigma(T)}(\gamma) > 0$. Consider the graph Γ_T of the restricted map $\Sigma(T) : Axis(\gamma) \to T$. Define the *horizontal hull* of Γ_T, denoted $Hull(\Gamma_T)$, to be the smallest subset H of $Axis(\gamma) \times T$ containing Γ_T such that $Axis(\gamma) \times \{y\} \cap H$ is an interval for every $y \in T$. Notice the following.

• $Hull(\Gamma_T)$ is foliated by horizontal lines $Axis(\gamma) \times \{y\} \cap Hull(\Gamma_T)$.

• $Hull(\Gamma_T)$ is invariant under the diagonal $\mathbb{Z}$-action induced by γ, and the quotient $K(\gamma, T) = Hull(\Gamma_T)/\mathbb{Z}$ is a finite, foliated 2-complex homotopy equivalent to the circle. The length with respect to this foliation of the generator of $\pi_1(K(\gamma, T))$ is $l_T(\gamma)$.

Now define $\Phi(T)$ to be the dual to the foliated complex $L(\gamma, T)$ obtained from the rose $\sigma(T)$ by attaching $K(\gamma, T)$ along $\Gamma_T/\mathbb{Z}$. The complex $L(\gamma, T)$ resolves T, and its dual, $\Phi(T)$, is resolved by $\sigma(T)$.

In general, $\Phi(T)$ is the dual to the foliated complex obtained from the rose $\sigma(T)$ by attaching $K(\gamma_i, T)$ for those $i = 1, \cdots, m$ for which $l_{\sigma(T)}(\gamma_i) > 0$.

The techniques of Skora [Sk1] show that Φ is continuous. A helpful observation is that $K(\gamma, T)$ can be given the structure of a band complex, with the number of bands bounded by a constant times the word length of γ. The rest follows easily. $\square$

Corollary 7.10. *For every open cover $\mathcal{U}$ of $\overline{\mathcal{X}}_n$, for every compact P, and for every map $f : P \to \overline{\mathcal{X}}_n$, there exists a map $f' : P \to \overline{\mathcal{X}}_n^{geom}$ so that f and f' are $\mathcal{U}$-close.*

Proof. Since P is compact, there exist $\gamma_1, \cdots, \gamma_m \in F_n$ such that if $T \in \overline{\mathcal{X}}_n$, $p \in P$, and $l_T(\gamma_i) = l_{f(p)}(\gamma_i)$ for $i = 1, \cdots, m$, then T and $f(p)$ are $\mathcal{U}$-close. Define $f' = \Phi \circ f$. $\square$

Corollary 7.11. *Every compact subset P of $\overline{\mathcal{X}}_n$ has dimension less than or equal to $3n-3$.*

Proof. Recall [HW, Corollary of Theorem V.9] that the dimension of a compact space Y is $\leq m$ provided that for every open cover $\mathcal{V}$ of Y, there is a map $g : Y \to Z$ such that $dim(Z) \leq m$ and the point preimages of g refine $\mathcal{V}$. Apply this to $Y = P$, $Z = \overline{\mathcal{X}}_n^{geom}$, and g a map approximating the inclusion of P into $\overline{\mathcal{X}}_n$. $\square$

Proof of Theorem 7.2. The space $\overline{\mathcal{X}}_n$ is a countable union of compact sets of dimension $\leq 3n-3$. Hence, [HW, Theorem III.2] $\overline{\mathcal{X}}_n$ has dimension $\leq 3n-3$. Since a generic graph of rank n has $3n-3$ edges, $\overline{\mathcal{X}}_n$ contains a subset of dimension $3n-3$. So, $dim(\overline{\mathcal{X}}_n) = 3n-3$. $\square$

Corollary 7.12. $dim(\overline{\mathcal{Y}}_n) = 3n-4$.

Proof. Let $\alpha \in F_n$. Let $U = U_\alpha$ be the subset of $\overline{\mathcal{Y}}_n$ where α is hyperbolic. The natural map $\overline{\mathcal{X}}_n \to \overline{\mathcal{Y}}_n$ is a trivial line bundle over U (take as a section the actions where the length of α is one). The space $\overline{\mathcal{Y}}_n$ is a countable union of such σ-compact sets U each of which has dimension $\leq 3n-4$ since crossing with $\mathbb{R}$ increases the dimension by one [HW, Remark after Theorem III 4]. $\square$

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