Homework 3: Cauchy, Morera, Integrals, Runge

Cauchy inequalities, Liouville

1. Suppose that f and g are entire functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there is a complex number λ such that $f(z) = \lambda g(z)$ for all $z \in \mathbb{C}$. Warning: If you consider f/g you should argue that it is well-defined at the zeros of g.

2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence ≥ 1. Suppose that $|f'(z)| \leq 1$ for all z with $|z| < 1$. Prove that $|a_n| \leq \frac{1}{n}$ for all n. By examples, show that these inequalities are sharp. Hint: Consider $g = f'$ and apply Cauchy inequalities on the circle of radius $1 - \epsilon$ and then let $\epsilon \to 0$. The examples are one for each n separately, not one for all n.

Morera's theorem.

3. Prove the following version of Morera's theorem. Suppose $f : \Omega \to \mathbb{C}$ is continuous and Ω is the open rectange $\{z \in \mathbb{C} \mid Re(z) \in (-1, 1), Im(z) \in (-1, 1)\}$. Suppose that $\int_\gamma f(z)dz = 0$ for every rectangle γ in Ω with sides parallel to the real and imaginary axes. Show that f is holomorphic. Note: The assumption that Ω is a rectangle is just for convenience; the statement is true for any open set. Hint: Construct a primitive just like in the triangle version.

Integrals.

In the first three problems use the same curve we used in class, consisting of segments $[-R, -\epsilon]$, $[\epsilon, R]$ and the two semicircles centered at 0 of radius ϵ and R. Recall that $\int_0^\infty$ by definition is $\lim_{\epsilon \to 0+, R \to \infty} \int_\epsilon^R$.

4. (Dirichlet integral) $\int_0^\infty \frac{\sin x}{x} dx = \pi/2$. Hint: e^{iz}/z. There is an additional trick here. You will have to subdivide the semicircle between R and $-R$ into 3 subarcs (two small ones near R and $-R$) and apply the Estimation Theorem separately on them.

5. $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi/2$. Hint: $\frac{1-e^{2iz}}{z^2}$. Actually, this integral is equivalent to the one we did in class after a simple substitution.

6. Prove that $\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{8}$. Hint: $\frac{3e^{iz} - e^{3iz}}{z^3}$.

7. Compute the Fresnel integrals

$$\int_0^\infty \cos(t^2)dt = \int_0^\infty \sin(t^2)dt = \sqrt{\frac{\pi}{8}}$$

You can use the Gaussian integral calculation

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

from real analysis. Hint: Consider $f(z) = e^{-z^2}$ and integrate on the sector that consists of segments $[0, R]$, $[0, Re^{i\pi/4}]$ and the circular arc connecting R and $Re^{i\pi/4}$. As in Problem 4 you will have to subdivide the circular arc.

Runge's theorem

8. In class we constructed a sequence of polynomials that pointwise converges to a discontinuous function. Find a variant of this construction to show that there is a sequence of polynomials that pointwise converges to the zero function on $\mathbb{C}$, but not uniformly in any neighborhood of 0. Comment: We will see later that if a sequence of holomorphic functions converges pointwise to f and all the functions are uniformly bounded on every compact set, then the convergence is uniform on compact sets and f is holomorphic.