Here is a solution to #35: If A is a strictly upper triangular $n \times n$ matrix, then $A^n = 0$.

Strictly upper triangular means that entries a_{ij} satisfy

$$a_{ij} = 0 \text{ when } i - j \geq 0$$

Most of you noticed that higher powers will necessarily have more and more diagonals with zeros. Formally, we make

Claim. For $r = 1, 2, \cdots, n$ the entries b_{ij} of A^r satisfy

$$b_{ij} = 0 \text{ when } i - j \geq 1 - r$$

When $r = 1$ this is true by assumption on A, and when we prove it for $r = n$ it will follow that $A^n = 0$ since *every* entry b_{ij} of an $n \times n$ matrix satisfies $i - j \geq 1 - n$.

It remains to prove the Claim by induction. For the inductive step $A^{r+1} = A^r A$, denote the entries of A^{r+1} by c_{ik}, the entries of A^r by b_{ij} and the entries of A by a_{jk}. Thus

$$c_{ik} = \sum_{j=1}^{n} b_{ij} a_{jk}$$

and we are assuming $a_{jk} = 0$ when $j - k \geq 0$ and $b_{ij} = 0$ when $i - j \geq 1 - r$. We will show that when $i - k \geq 1 - (r+1) = -r$ then every summand in the formula for c_{ik} is 0. For this, we only have to see that for every j either $j - k \geq 0$ or $i - j \geq 1 - r$. So suppose that both of these fail, i.e. $j - k \leq -1$ and $i - j \leq -r$. Adding up gives $i - k \leq -1 - r$ contradicting the assumption $i - k \geq -r$.