TAYLOR CONDITIONS OVER FINITE FIELDS

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Abstract. In [Poo04], Poonen proves a Bertini theorem over finite fields, including prescribing the first few Taylor coefficients of sections at finitely many points. In the motivic setting, [BH21] proved an analogous result but allowing much more general Taylor conditions. We extend Poonen's result in the arithmetic setting to Taylor conditions arising as subsheaves of the sheaf of differentials such that the corresponding quotient is locally free.

Contents

1. Introduction

For X a smooth quasiprojective subscheme of $\mathbb{P}^n$ over a finite field $\mathbb{F}_q$, Poonen showed in [Poo04] the existence of smooth hypersurface sections of X and computed the density of smooth hypersurface sections in all hypersurface sections to be $\zeta_X(\dim X + 1)^{-1}$, where ζ_X is the zeta function of X. He also allowed for prescribing the first few coefficients of the Taylor expansions of hypersurfaces at finitely many points. It is natural to extend the problem to more general conditions on the Taylor expansions. As far as the author knows, questions like the following are not within the scope of Poonen's theorem or its existing generalizations which include [BK12], [EW15], [GK23], [Gun17], [Poo08], and [Wut14].

Question 1. Let $\operatorname{char}(\mathbb{F}_q) \neq 2$. Choose four closed points of $\mathbb{P}^2_{\mathbb{F}_q}$ that are geometrically in general position. Let $i : X \hookrightarrow \mathbb{P}^2_{\mathbb{F}_q}$ be a curve whose geometric points are in general position with these four points. For each closed point $x \in X$, there is a unique conic C_x passing through the four points and X. What is the probability that a random plane curve intersects C_x transversely at x for each closed point $x \in X$?

This question is answered in Example 5.3 and requires considering Taylor conditions arising from subsheaves of the sheaf of differentials. Such Taylor conditions are addressed in the following theorem which is the main result of this paper. See Section 2.1 for the definition of the sheaf of principal parts $\mathscr{P}^1$.

Theorem 1.1. *Let $i : X \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$ be a quasiprojective subscheme of dimension m. Let $\mathscr{Q}$ be a locally free quotient of $i^*\Omega^1_{\mathbb{P}^n}$ of rank $\ell \geq m$, and let $\mathscr{K}$ denote the kernel of $i^*\Omega^1_{\mathbb{P}^n} \twoheadrightarrow \mathscr{Q}$. For each d, define*

$$\mathscr{E}_d := (i^*\mathscr{P}^1(\mathscr{O}_{\mathbb{P}^n}(d)))/\mathscr{K}(d)$$

where we view $\mathscr{K}(d)$ as a subsheaf of $i^*\mathscr{P}^1(\mathscr{O}_{\mathbb{P}^n}(d))$ via the exact sequence

$$0 \to i^*\Omega^1_{\mathbb{P}^n}(d) \to i^*\mathscr{P}^1(\mathscr{O}_{\mathbb{P}^n}(d)) \to \mathscr{O}_X(d) \to 0$$

At each closed point $x \in \mathbb{P}^n$, this defines a 1-infinitesimal Taylor condition $\mathcal{T}_{d,x} \subset \mathscr{O}_{\mathbb{P}^n}(d)_x/\mathfrak{m}_x^2$ at x given by not vanishing in the fiber of $\mathscr{E}_d$ at x. By convention, $\mathcal{T}_d$ is always satisfied if $x \notin X$.

Then

$$\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{T}_d) = \prod_{\text{closed } x\in X} (1 - q^{-(\ell+1)\deg(x)}) = \zeta_X(\ell+1)^{-1}.$$

Note that for X smooth, taking $\mathscr{Q} = \Omega^1_X$ recovers Poonen's Bertini theorem.

Theorem 1.1 is motivated by the significantly more general Taylor conditions considered by [BH21] in the motivic setting, i.e., in the Grothendieck ring of varieties. There the authors ask if an arithmetic analog of the following theorem holds over $\mathbb{F}_q$ (see the paper for notation):

Theorem ([BH21, Theorem B])**.** *Fix $f : X \to S$, a proper map of varieties over a field K, $\mathscr{F}$ a coherent sheaf on X, $\mathscr{L}$ a relatively ample line bundle on X, and $r, M \geq 0$. Then, there is an $\epsilon > 0$ such that as T ranges over all r-infinitesimal Taylor conditions on $\mathscr{F}(d) = \mathscr{F} \otimes \mathscr{L}^d$ with M-admissible complement,*

$$\frac{[\mathbb{V}(f_*\mathscr{F}(d))^{T\text{-everywhere}}]}{[\mathbb{V}(f_*\mathscr{F}(d))]} = \prod_{x\in X/S} \left(1 - \frac{[T^c]_x}{[\mathbb{V}(\mathscr{P}^r_{/S}\mathscr{F}(d))]_x} t\right)\Bigg|_{t=1} + O(\mathbb{L}^{-\epsilon d})$$

in $\widehat{\widetilde{\mathcal{M}}}_X$.

For Bilu and Howe, a Taylor condition is just a constructible subset of the sheaf of principal parts (viewed as a scheme) and the M-admissible condition ensures the motivic Euler product converges. In the arithmetic setting, we also need a good notion of "admissibility" for a Taylor condition such that the probability that the condition is satisfied everywhere factors into the local probabilities at closed points. A counterexample to the most general such Taylor conditions is given in Example 3.1, suggesting more structure, possibly algebraic as in Theorem 1.1, is necessary.

This paper will set up our notation and give some properties of the sheaf of principal parts in Section 2.1, provide a counterexample to the most general Taylor conditions in Section 3, prove Theorem 1.1 in Section 4, and give some applications in Section 5.

While this paper only deals with the case where $S = \operatorname{Spec} \mathbb{F}_q$, $\mathscr{F} = \mathscr{O}_X$, and $r = 1$, our definitions below point toward the level of generality we feel is possible.

2. Notation and definitions

Throughout, let q be a prime power and $\mathbb{F}_q$ the field with q elements. Let $S = \mathbb{F}_q[x_0, \dots, x_n]$ and identify $H^0(\mathbb{P}^n_{\mathbb{F}_q}, \mathscr{O}(d))$ with degree d homogeneous polynomials S_d in S. Let $A = \mathbb{F}_q[x_1, \dots, x_n]$ and $A_{\leq d}$ the polynomials in A of degree at most d.

Definition. Let $\mathscr{F}$ be a coherent sheaf on a variety X over $\mathbb{F}_q$. An *r-infinitesimal Taylor condition on $\mathscr{F}$ at a closed point $x \in X$* is a subset

$$\mathcal{T}_x \subseteq \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x}/\mathfrak{m}_x^{r+1} =: \mathscr{F}|_{x^{(r)}}.$$

An *r-infinitesimal Taylor condition $\mathcal{T}$ on $\mathscr{F}$* is a choice of an r-infinitesimal Taylor condition $\mathcal{T}_x$ at x on $\mathscr{F}$ for each closed point x.

We say that a global section $s \in H^0(X, \mathscr{F})$ *satisfies $\mathcal{T}$ at $x \in X$* if its image in $\mathscr{F}|_{x^{(r)}}$ lies in $\mathcal{T}_x$, and *satisfies* $\mathcal{T}$ if it satisfies $\mathcal{T}$ at every closed point $x \in X$.

Definition. Let $\mathscr{F}$ be a coherent sheaf on a variety X over $\mathbb{F}_q$. For a subset $\mathcal{P}$ of the finite dimensional $\mathbb{F}_q$-vector space $H^0(X, \mathscr{F})$, denote by $\operatorname{Prob}(s \in \mathcal{P})$ the probability that a random uniformly distributed global section s of $\mathscr{F}$ belongs to $\mathcal{P}$, i.e.,

$$\operatorname{Prob}(s \in \mathcal{P}) := \frac{\#\mathcal{P}}{\#H^0(X, \mathscr{F})}.$$

Remark 2.1. The definition above differs from that of [EW15]. When they write $\operatorname{Prob}(s \in \mathcal{P})$, they mean (in our notation) $\lim_{d\to\infty} \operatorname{Prob}(s_d \in \mathcal{P}_d)$ where for each $d \geq 0$, $\mathcal{P}_d \subseteq H^0(X, \mathscr{F}(d))$ and s_d is a uniform random global section of $\mathscr{F}(d)$.

2.1. **Sheaves of principal parts.** We recall the definition of the sheaf of principal parts and collect some of its relevant properties.

Definition. Let $X \to S$ be a morphism of schemes and $\mathscr{F}$ an $\mathscr{O}_X$-module. Let $\Delta^{(r)}$ be the r-th infinitesimal neighborhood of the diagonal Δ in $X \times_S X$ and let $\delta^{(r)} : \Delta^{(r)} \to X \times_S X$ be the canonical morphism. Denote by $\pi_1, \pi_2 : X \times_S X \to X$ the corresponding projections and set $p = \pi_1 \circ \delta^{(r)}$ and $q = \pi_2 \circ \delta^{(r)}$. The *sheaf of r-th order principal parts of $\mathscr{F}$ on X over S* is

$$\mathscr{P}^r_{X/S}(\mathscr{F}) := p_*(q^*\mathscr{F}).$$

By definition this is an $\mathscr{O}_X$-module. If $\mathscr{F} = \mathscr{O}_X$, we write $\mathscr{P}^r_{X/S} := \mathscr{P}^r_{X/S}(\mathscr{O}_X)$.

References given below are not necessarily the original source of the result.

Lemma 2.2 ([Gro67, Proposition 16.7.3])**.** *If $\mathscr{F}$ is quasi-coherent (resp. coherent, of finite type, of finite presentation), then $\mathscr{P}^r_{X/S}(\mathscr{F})$ is quasi-coherent (resp. coherent, of finite type, of finite presentation).*

Lemma 2.3 ([Gro67, Corollary 16.4.12] and [Ben70, III, Lemma 2.1 and Proposition 2.2])**.** *If $S = \operatorname{Spec} k$ for k a field, $\mathscr{F}$ is quasi-coherent, and $x \in X$ is rational over k, then the fiber $\mathscr{P}^r_{X/S}(\mathscr{F})|_x = \mathscr{P}^r_{X/S}(\mathscr{F})_x \otimes_{\mathscr{O}_{X,x}} \kappa(x)$ is canonically isomorphic to $\mathscr{F}_{X,x}/\mathfrak{m}_x^{r+1}$.*

If k is perfect, then the same is true for any closed point $x \in X$.

Remark 2.4. In our notation, Lemma 2.3 says that an r-infinitesimal Taylor condition on $\mathscr{F}$ is just a subset of the fiber of $\mathscr{P}^r_{X/k}(\mathscr{F})$ at x.

Lemma 2.5 ([Per95, A, Proposition 3.4])**.** *If $X \to S$ is differentially smooth (see* [Gro67, 16.10]*), and $\mathscr{F}$ is locally free on X, then there is an exact sequence of $\mathscr{O}_X$-modules*

$$0 \longrightarrow \operatorname{Sym}^r_{\mathscr{O}_X}(\Omega^1_{X/S}) \otimes_{\mathscr{O}_X} \mathscr{F} \longrightarrow \mathscr{P}^r_{X/S}(\mathscr{F}) \longrightarrow \mathscr{P}^{r-1}_{X/S}(\mathscr{F}) \longrightarrow 0.$$

If X, Y are smooth S-schemes, $f : X \to Y$ is a morphism of S-schemes, and $\mathscr{G}$ is locally free on Y, then there is a map of exact sequences of $\mathscr{O}_X$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Sym}^r_{\mathscr{O}_X}(f^*\Omega^1_{Y/S}) \otimes_{\mathscr{O}_X} f^*\mathscr{G} & \longrightarrow & f^*\mathscr{P}^r_{Y/S}(\mathscr{G}) & \longrightarrow & f^*\mathscr{P}^{r-1}_{Y/S}(\mathscr{G}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Sym}^r_{\mathscr{O}_X}(\Omega^1_{X/S}) \otimes_{\mathscr{O}_X} f^*\mathscr{G} & \longrightarrow & \mathscr{P}^r_{X/S}(f^*\mathscr{G}) & \longrightarrow & \mathscr{P}^{r-1}_{X/S}(f^*\mathscr{G}) & \longrightarrow & 0 \end{array}$$

Corollary 2.6 ([Per95, A, Proposition 3.3])**.** *In the setting of Lemma 2.5, if $\mathscr{F}$ is locally free of rank n, then $\mathscr{P}^r_{X/S}(\mathscr{F})$ is locally free of rank $n \cdot \binom{\dim X + r}{r}$.*

3. Counterexamples to most general Taylor conditions

The following example shows that arbitrary set-theoretic constructions of Taylor conditions can produce local probabilities whose product is not the global probability of the condition being satisfied.

Example 3.1 (Diagonal argument)**.** Let $X = \mathbb{P}^n_{\mathbb{F}_q}$ and $\mathscr{F} = \mathscr{O}_{\mathbb{P}^n}$. Both the union of global sections S_d over all $d \in \mathbb{N}$ and the set of closed points of $\mathbb{P}^n$ are countably infinite; let $f_1, f_2, \ldots$ and $x_1, x_2, \ldots$ be enumerations of them, respectively. For every closed point x, fix an isomorphism $\mathscr{O}_{\mathbb{P}^n}(d)|_{x^{(1)}} \cong \mathscr{O}_{\mathbb{P}^n}|_{x^{(1)}}$. Define a 1-infinitesimal Taylor condition as follows: for each i, set $\mathcal{T}_{d,x_i}$ to be all of $\mathscr{O}_{\mathbb{P}^n}|_{x_i^{(1)}}$ except the Taylor expansion of f_i (this does not depend on d). Then the local probabilities are $p_{x_i} = 1 - q^{-\deg(x_i)(n+1)}$, and the product over all closed points is $\zeta_{\mathbb{P}^n}(n+1)^{-1}$. But by construction, no section can satisfy this Taylor condition at all closed points.

Some algebraic nature to the condition is likely necessary in general. In Theorem 1.1, this manifests as "locally free quotients of the sheaf of principal parts".

4. More general Taylor conditions

We now use Poonen's method of the closed point sieve to prove Theorem 1.1.

4.1. **Singular points of low degree.** The following lemma says that for finitely many closed points, the local probabilities are independent.

Lemma 4.1 (Singularities of low degree)**.** *Let X, $\mathcal{Q}$, and $\mathcal{T}_d$ be as in Theorem 1.1. For $e > 0$, define*

$$\mathcal{P}^{\mathrm{low}}_{e,d} := \{f \in S_d \mid f \textit{ satisfies } \mathcal{T}_d \textit{ at all } x \textit{ with } \deg(x) < e\}.$$

Let $X_{<e}$ be the closed points of X of degree less than e. Then

$$\lim_{d\to\infty} \mathrm{Prob}(f \in \mathcal{P}^{\mathrm{low}}_{e,d}) = \prod_{x \in X_{<e}} \left(1 - q^{-(\ell+1)\deg(x)}\right).$$

Proof. Let $X_{<e} = \{x_1, \ldots, x_s\}$. By definition, $f \in S_d$ fails $\mathcal{T}_d$ at x_i if and only if it vanishes under the composition

$$S_d \to \mathscr{O}_{\mathbb{P}^n}(d)_{x_i}/\mathfrak{m}^2_{x_i} \to \mathscr{E}_d|_{x_i}$$

for some $i \in \{1, \ldots, s\}$. Thus $\mathcal{P}^{\mathrm{low}}_{e,d}$ consists of the preimage of $\prod_{i=1}^s (\mathscr{E}_d|_{x_i} \setminus \{0\})$ under the composition

$$S_d \to \prod_{i=1}^s \mathscr{O}_{\mathbb{P}^n}(d)_{x_i}/\mathfrak{m}^2_{x_i} \to \prod_{i=1}^s \mathscr{E}_d|_{x_i}$$

This first map is surjective for $d \gg 1$ by [Poo04, Lemma 2.1] and the second since $i^*\mathscr{P}^1(\mathscr{O}_{\mathbb{P}^n}(d)) \to \mathscr{E}_d$ is surjective, so the composition is surjective.

We have a filtration of $\kappa(x_i)$-vector spaces $0 \subset \mathcal{Q}(d)|_{x_i} \subset \mathscr{E}_d|_{x_i}$ whose quotients have dimensions 1 and ℓ, respectively, hence $\mathscr{E}_d|_{x_i} \setminus \{0\}$ has size $q^{(\ell+1)\deg(x_i)} - 1$, and the local probability of vanishing is $1 - q^{-(\ell+1)\deg(x_i)}$. As this does not depend on d, the result follows. □

4.2. **Singular points of medium degree.**

Lemma 4.2 (Singularities of medium degree)**.** *Let X, $\mathcal{Q}$, and $\mathcal{T}_d$ be as in Theorem 1.1. For $e > 0$, define*

$$\mathcal{Q}^{\mathrm{med}}_{e,d} := \{f \in S_d \mid f \text{ fails } \mathcal{T}_d \text{ at some } x \text{ with } e \le \deg(x) \le \tfrac{d}{\ell+1}\}.$$

Then

$$\lim_{e\to\infty} \lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{Q}^{\mathrm{med}}_{e,d}) = 0.$$

Proof. Let x be a closed point with $e \le \deg(x) \le \ell+1$. We have $\dim_{\mathbb{F}_q} \mathscr{E}_d|_x = (\ell+1)\deg(x) \le d$ by assumption. Note the argument in [Poo04, Lemma 2.1] works exactly the same here with the map $S_d \to \mathscr{E}_d|_x$, so this map is surjective and identical reasoning as in [Poo04, Lemma 2.3] shows the fraction of $f \in S_d$ that vanish in $\mathscr{E}_d|_x$ is $q^{-(\ell+1)\deg(x)}$.

Now we follow Poonen's proof of [Poo04, Lemma 2.4]. By [LW54], there is a constant $c > 0$ depending only on X such that $\#X(\mathbb{F}_{q^r}) \le cq^{rm}$. With the result above, this gives

$$\begin{aligned}
\operatorname{Prob}(f \in \mathcal{Q}^{\mathrm{med}}_{e,d}) &\le \sum_{r=e}^{\lfloor d/(m+1)\rfloor} (\#\text{ of points of degree } r)\cdot q^{-(\ell+1)r} \\
&\le \sum_{r=e}^{\lfloor d/(m+1)\rfloor} \#X(\mathbb{F}_{q^r})\cdot q^{-(\ell+1)r} \\
&\le \sum_{r=e}^{\infty} cq^{rm}q^{-(\ell+1)r}
\end{aligned}$$

Since $\ell \ge m$, this converges to $\frac{cq^{e(m-\ell-1)}}{1-q^{m-\ell-1}}$. This is independent of d and goes to zero as e goes to ∞. $\square$

4.3. **Singular points of high degree.** As usual with proofs using the closed point sieve, showing the contribution from high degree points is negligible is the hardest part of the proof.

Lemma 4.3. *Let X, $\mathcal{Q}$, and $\mathcal{T}_d$ be as in Theorem 1.1. Define*

$$\mathcal{Q}^{\mathrm{high}}_{d} := \{f \in S_d \mid f \text{ fails } \mathcal{T}_d \text{ at some } x \text{ with } \deg(x) > \tfrac{d}{\ell+1}\}.$$

Then $\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{Q}^{\mathrm{high}}_d) = 0$.

Proof. As in [Poo04, Lemma 2.6], we reduce to the affine case $i : X \hookrightarrow \mathbb{A}^n$, also dehomogenizing to identify S_d with $A_{\le d}$.

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & i^*\Omega^1_{\mathbb{A}^n/\mathbb{F}_q} \arrow[r, hook] \arrow[d] \arrow[dr, dashed, "F"] & i^*\mathscr{P}^1(\mathcal{O}_{\mathbb{A}^n}) \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{Q} \arrow[r, hook] & \mathscr{E} & &
\end{tikzcd}$$

Given a closed point $x \in X$, choose elements $t_1, \dots, t_\ell \in A$ such that the images under F of the pullbacks of $dt_1, \dots, dt_\ell$ form a basis for $(\operatorname{im} F)_x \subset \mathscr{E}_x$

over $\mathscr{O}_{X,x}$ (recall $\mathscr{E}$ has rank $\ell+1$). Call these basis elements $Q_1, \dots, Q_\ell$. Let $\partial_1, \dots, \partial_\ell$ be the corresponding dual basis.

Now we mimic Poonen's proof of [Poo04, Lemma 2.6].

We have

$$\begin{aligned}\operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{E}_x, \mathscr{O}_{X,x}) &\subset \operatorname{Hom}_{\mathscr{O}_{X,x}}((i^*\Omega^1_{\mathbb{A}^n})_x, \mathscr{O}_{X,x})\\ &= \operatorname{Hom}_{\mathscr{O}_{\mathbb{A}^n,x}}(\Omega^1_{\mathbb{A}^n,x}, \mathscr{O}_{X,x})\\ &= \operatorname{Der}_{\mathbb{F}_q}(\mathscr{O}_{\mathbb{A}^n,x}, \mathscr{O}_{X,x})\end{aligned}$$

Thus we can think of the dual basis elements ∂_i as $\mathbb{F}_q$-derivations $\mathscr{O}_{\mathbb{A}^n,x} \to \mathscr{O}_{X,x}$. Choose $s \in A/I(X)$ with $s(x) \neq 0$ to clear denominators so $D_i = s\partial_i$ is a global derivation $A \to A/I(X)$. We can find a neighborhood N_x of x on which $Q_1, \dots, Q_\ell$ generate $\operatorname{im} F$ and such that $s \in \mathscr{O}_X(N_x)^\times$. As we can cover X with finitely many such N_x, we may assume $X \subset N_x$, and that the $Q_1, \dots, Q_\ell$ generate $\operatorname{im} F$ globally.

Set $\tau = \max_i\{\deg t_i\}$, $\gamma = \lfloor (d-\tau)/p \rfloor$, and $\eta = \lfloor d/p \rfloor$. If $f_0 \in A_{\le d}$, $g_1, \dots, g_\ell \in A_{\le \gamma}$, and $h \in A_{\le \eta}$ are selected uniformly at random, then the distribution of

$$f = f_0 + g_1^p t_1 + \dots + g_\ell^p t_\ell + h^p$$

is uniform over $A_{\le d}$. We'll bound the probability that for such an f, there's a closed point $y \in X_{>d/(\ell+1)}$ where f is zero in the fiber of $\mathscr{E}$ at y; equivalently, when $(D_1 f)(y) = \dots = (D_\ell f)(y) = 0$.

We have $D_i f = D_i f_0 + g_i^p s$ for $i = 1, \dots, \ell$. By abuse of notation we'll consider the $D_i f$ as defining hypersurfaces in $\mathbb{A}^n$ by choosing a lift to A of minimal degree. Define

$$W_i = X \cap \{D_1 f = \dots = D_i f = 0\}.$$

Claim 1. *For $0 \le i \le \ell-1$, conditioned on a choice of $f_0, g_1, \dots, g_i$ such that $\dim(W_i) \le m-i$, the probability that $\dim(W_{i+1}) \le m-i-1$ is $1-o(1)$ as $d \to \infty$.*

Let $V_1, \dots, V_e$ be the $(m-i)$-dimensional irreducible components of $(W_i)_{\mathrm{red}}$. By Bézout's theorem,

$$e \le (\deg \overline{X})(\deg D_1 f) \cdots (\deg D_i f) = O(d^i)$$

as $d \to \infty$, where $\overline{X}$ is the projective closure of X. As $\dim V_k \ge 1$, there exists a coordinate x_j, depending on k, such that the projection $x_j(V_k)$ has dimension 1.

We want to bound the set

$$G_k^{\mathrm{bad}} := \{g_{i+1} \in A_{\le\gamma} \mid D_{i+1} f = D_{i+1} f_0 + g_{i+1}^p s \text{ vanishes identically on } V_k\}$$

since for any $g_{i+1} \in G_k^{\mathrm{bad}}$, $V_k \subset W_{i+1}$ and then $\dim(W_{i+1})$ would fail to be $\le m-i-1$.

If $g, g' \in G_k^{\mathrm{bad}}$, then on V_k,

$$\begin{aligned} 0 &= \frac{g^p s - g'^p s}{s} \\ &= g^p - g'^p \\ &= (g-g')^p \end{aligned}$$

so if G_k^{bad} is nonempty, it is a coset of the subspace of functions in $A_{\le\gamma}$ that vanish on V_k. The codimension of that subspace is at least $\gamma+1$ since a nonzero polynomial in x_j alone does not vanish on V_k. Thus the probability that $D_{i+1}f$ vanishes on some V_k is at most $eq^{-(\gamma+1)} = o(1)$ as $d \to \infty$.

Claim 2. *Conditioned on a choice of $f_0, g_1, \dots, g_\ell$ for which W_ℓ is finite,* $\mathrm{Prob}(H_f \cap W_\ell \cap X_{>d/(\ell+1)} = \emptyset) = 1 - o(1)$ *as $d \to \infty$.*

In fact, we need only show this for $H_f \cap W_m \cap X_{>d/(\ell+1)}$. The same Bézout argument as above shows $\#W_m$ is $O(d^m)$. For a given $y \in W_m$, the set H^{bad} of $h \in A_{\le\eta}$ for which H_f passes through y is either empty or a coset of $\ker(\mathrm{eval}_y : A_{\le\eta} \to \kappa(y))$.

If $\deg(y) > \frac{d}{\ell+1}$, then [Poo04, Lemma 2.5] implies $\frac{\#H^{\mathrm{bad}}}{\#A_{\le\eta}} \le q^{-\nu}$ where $\nu = \min(\eta+1, \frac{d}{\ell+1})$. Hence

$$\mathrm{Prob}(H_f \cap W_m \cap X_{>d/(\ell+1)} = \emptyset) \le \#W_m q^{-\nu} = O(d^m q^{-\nu})$$

which by assumption is $o(1)$ as $d \to \infty$.

Given the two claims, we have

$$\begin{aligned} &\lim_{d\to\infty} \mathrm{Prob}\big(\dim W_i = m - i \text{ and } H_f \cap W_m \cap X_{>d/(\ell+1)} = \emptyset\big) \\ &\qquad = \prod_{i=0}^{m-1} (1 - o(1)) \cdot (1 - o(1)) \\ &\qquad = 1 - o(1) \end{aligned}$$

So the same holds for W_ℓ. But now $H_f \cap W_\ell$ is the subvariety of X defined by failing $\mathcal{T}_d$, so $H_f \cap W_\ell \cap X_{>d/(\ell+1)}$ is the set of points of degree $> \frac{d}{\ell+1}$ where $H_f \cap X$ fails $\mathcal{T}_d$. □

4.4. **Proof of Theorem 1.1.**

Proof. We have

$$\mathcal{T}_d \subseteq \mathcal{P}_{e,d}^{\mathrm{low}} \subseteq \mathcal{T}_d \cup \mathcal{Q}_{e,d}^{\mathrm{med}} \cup \mathcal{Q}_d^{\mathrm{high}}$$

so

$$\begin{aligned} \mathrm{Prob}(s \in \mathcal{P}_{e,d}^{\mathrm{low}}) &\ge \mathrm{Prob}(s \in \mathcal{T}_d) \\ &\ge \mathrm{Prob}(s \in \mathcal{P}_{e,d}^{\mathrm{low}}) - \mathrm{Prob}(s \in \mathcal{Q}_{e,d}^{\mathrm{med}}) - \mathrm{Prob}(s \in \mathcal{Q}_d^{\mathrm{high}}). \end{aligned}$$

By Lemmas 4.1 to 4.3, letting d, then e go to ∞ gives the result. □

5. Applications

Example 5.1 (Poonen's Bertini)**.** To get [Poo04, Theorem 1.1], assume X is smooth and take $\mathcal{Q} = \Omega^1_{X/\mathbb{F}_q}$ in Theorem 1.1.

Example 5.2. Let $i : X \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$ be a quasiprojective subscheme of dimension m and let $\Delta : X \hookrightarrow X \times_{\mathbb{F}_q} \mathbb{P}^n$ be the graph of i. Suppose $j : Z \hookrightarrow X \times_{\mathbb{F}_q} \mathbb{P}^n$ is a closed subscheme such that the projection $\varphi : Z \to X$ is smooth of relative dimension $\ell \geq m$, and such that Δ factors as

$$X \xrightarrow{\alpha} Z \overset{j}{\hookrightarrow} X \times \mathbb{P}^n$$

for some morphism $\alpha : X \to Z$.

We have a surjection of sheaves

$$\Omega^1_{X\times\mathbb{P}^n/X} \twoheadrightarrow j_*\Omega^1_{Z/X}$$

which induces a surjection

$$\Delta^*\Omega^1_{X\times\mathbb{P}^n/X} \twoheadrightarrow \Delta^* j_*\Omega^1_{Z/X}$$

The left side is isomorphic to $i^*\Omega^1_{\mathbb{P}^n}$; indeed, let $p : X \times \mathbb{P}^n \to \mathbb{P}^n$ be projection onto the second coordinate. Then by standard base change for the sheaf of differentials,

$$\begin{aligned}
\Delta^*\Omega^1_{X\times\mathbb{P}^n/X} &\cong \Delta^* p^*\Omega^1_{\mathbb{P}^n/\mathbb{F}_q} \\
&= (p\circ\Delta)^*\Omega^1_{\mathbb{P}^n/\mathbb{F}_q} \\
&= i^*\Omega^1_{\mathbb{P}^n/\mathbb{F}_q}.
\end{aligned}$$

Define $\mathcal{Q} = \Delta^* j_*\Omega^1_{Z/X}$. This is locally free: by assumption, $\Delta = j \circ \alpha$ so $\mathcal{Q} = \alpha^* j^* j_*\Omega^1_{Z/X} \cong \alpha^*\Omega^1_{Z/X}$. As φ is smooth of relative dimension ℓ, $\Omega^1_{Z/X}$ is locally free of rank ℓ and thus so is $\mathcal{Q}$.

Define $\mathcal{T}_d = \mathcal{Q}(d)|_x \setminus \{0\}$. Applying Theorem 1.1, we get

$$\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{T}_d) = \zeta_X(\ell+1)^{-1}.$$

Example 5.3. We now answer Question 1 as a specific instance of Example 5.2. Let $\operatorname{char}(\mathbb{F}_q) \neq 2$. Choose four closed points of $\mathbb{P}^2_{\mathbb{F}_q}$ that are geometrically in general position. Let $i : X \hookrightarrow \mathbb{P}^2_{\mathbb{F}_q}$ be a curve whose geometric points are in general position with these four points. Then for each closed point $x \in X$, there is a unique smooth conic C_x passing through the four points and X. Let $j : C \hookrightarrow X \times \mathbb{P}^2$ be the inclusion of the subscheme C parameterizing the data $\{(x, y) \mid x \in X, y \in C_x\}$. Then Δ factors as $j\circ\alpha$ where α is the diagonal into C, and $\varphi : C \to X$ is smooth of relative dimension 1, so the conditions of the example are satisfied.

Let $f \in S_d$. With $\mathcal{Q}$ defined as in Example 5.2, the hypersurface H_f intersects C_x transversely at x if and only if it does not vanish in the fiber of $\mathcal{Q}$ at x. Thus the example above shows the probability that a random plane curve intersects C_x transversely at x for all closed $x \in X$ is $\zeta_X(2)^{-1}$.

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