5. Construction of the Hilbert Scheme: We'll start with a discussion of Grassmannians, then construct Grothendieck's quot schemes and Hilbert schemes using the results of the previous two sections.

Fix a field k. The Grassmannian $G(m,n)$ over k is a projective k-scheme which we may define as follows. Start with the "wedge" map:

$$f : \mathrm{Spec}(k[x_{i,j}]) = \mathbf{A}_k^{mn} \to \mathbf{A}_k^{\binom{n}{m}} = \mathrm{Spec}(k[x_J])$$

where $1 \le i \le m$, $1 \le j \le n$ and $J = (j_1, ..., j_m)$ for $1 \le j_1 < ... < j_m \le n$ (we'll call J a sorted m-tuple). Then f is defined by $f^*(x_J) = \det(x_{i,j_l})$ the determinant of the square submatrix of $(x_{i,j})$ obtained by removing all columns except those indexed by J. Define $\phi : \mathbf{A}_k^{mn} - - > \mathbf{P}_k^{\binom{n}{m}-1}$ to be the composition of the morphism f with the rational map to projective space. Thus ϕ is a morphism when restructed to the Zariski open subset $U := \mathbf{A}_k^{mn} - V(\{\det(x_{i,j_l})\})$ of matrices of "full rank."

If $I = (i_1, ..., i_m)$ is an (unsorted) m-tuple of integers with $1 \le i_l \le n$ and if $\sigma \in \Sigma_m$ is a permutation, let $\sigma(I) = (i_{\sigma(1)}, ..., i_{\sigma(m)})$. Note that there is at most one sorted m-tuple J in each orbit of Σ_m and exactly one if the i_l are distinct. It follows that we obtain well-defined $x_I \in \mathrm{H}^0(\mathbf{P}_k^{\binom{n}{m}-1}, \mathcal{O}(1))$ by setting:

$$x_I = \begin{cases} 0 & \text{if the } i_l \text{ are not distinct} \\ (-1)^{\mathrm{sgn}(\sigma)} x_J & \text{if } \sigma(I) = J, \text{ a sorted multi-index} \end{cases}$$

The **Plücker quadric** associated to a pair I, J of m-tuples and an integer $1 \le s \le m$ is given by:

$$Q_{I,J,s} := x_I x_J - \sum_{l=1}^{m} x_{(i_1,...,i_{s-1},j_l,i_{s+1},...,i_m)} x_{(j_1,...,j_{l-1},i_s,j_{l+1},...,j_m)}$$

The **Grassmannian** $G(m,n) \subset \mathbf{P}_k^{\binom{n}{m}-1}$ is the zero scheme of the set of all the Plücker quadrics.

Remark: These quadrics are redundant. For example, if $m = 2, n = 4$, then

$$G(2,4) = V(x_{(1,2)}x_{(3,4)} - x_{(1,3)}x_{(2,4)} + x_{(1,4)}x_{(2,3)})$$

i.e. $G(2,4) \subset \mathbf{P}_k^5$ is a nonsingular quadric hypersurface.

Proposition: $\phi : U \to \mathbf{P}_k^{\binom{n}{m}-1}$ factors through $G(m,n)$ and:

(a) $\phi : U \to G(m,n)$ is a locally trivial fiber bundle, with fiber $\mathrm{GL}(m,k)$. In particular, ϕ determines a bijection from the set of m-dimensional subspaces of k^n to the set of k-rational points of $G(m,n)$.

(b) Let $U_{x_J} = \mathbf{P}_k^{\binom{n}{m}-1} - V(x_J)$. Then $G(m,n) \cap U_{x_J} \cong \mathbf{A}_k^{m(n-m)}$. Hence $G(m,n)$ is a nonsingular projective variety over k of dimension $m(n-m)$.

Proof: ϕ is equivariant with respect to the "left multiplication" action $\mu : \mathrm{GL}(m,k) \times U \to U$ where $\mathrm{GL}(m,k) \subset \mathbf{A}_k^{m^2}$ is the open subscheme of invertible $m \times m$ matrices. That is, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{GL}(m,k) \times U & \xrightarrow{\mu} & U \\ \downarrow \pi & & \downarrow \phi \\ U & \xrightarrow{\phi} & \mathbf{P}_k^{\binom{n}{m}-1} \end{array}$$

This follows from the fact that if we multiply an $m \times n$ matrix A on the left by a square matrix M, then the determinants of the $m \times m$ submatrices of A are all multiplied by $\det(M)$.

Fix J and consider the closed subscheme $\mathbf{A}_k^{m(n-m)} \cong Z_J \subset U$ of $m \times n$ matrices whose square submatrix corresponding to J is the identity. Then the following are easily verified:

(i) $\mu :\ \mathrm{GL}(m,k) \times Z_J \to \phi^{-1}(U_{x_J})$ is an isomorphism.

(ii) $\phi^*(Q_{K,L,s}/x_J^2) \in I(Z_J)$ for all the Plücker quadrics.

(iii) $\phi|_{Z_J} : Z_J \to G(m,n) \cap U_{x_J}$ is an isomorphism.

The proposition now follows quickly. By (i) and (ii), and the fact that ϕ is μ-equivariant, it follows that $\phi^*(Q_{I,J,s}/x_J^2) = 0 \in I(\phi^{-1}U_{x_J})$ hence that ϕ factors through the Grassmannian.

Next, (i) and (iii) imply that $\phi : U \to G(m,n)$ is a locally trivial fibration. And again using the action, we see that given a k-rational point $x \in G(m,n)$, the closed points in the fiber $\phi^{-1}(x)$ are just the orbit of an $m \times n$ matrix A under the left action by $\mathrm{GL}(m,k)$. If we write the rows of A as vectors $v_1, ..., v_m$, then this identifies x with the subspace of k^n generated by $v_1, ..., v_m$, giving the bijection. This completes the proof of the Proposition.

Given a scheme T, consider exact sequences: $(*)\ 0 \to K \to \mathcal{O}_T^n \to Q \to 0$ where K, Q are locally free sheaves on T of ranks m and $n-m$, respectively. There is a natural equivalence relation on these. Namely, we'll say two such sequences $(*)$ and $(*')$ are equivalent if there is a commuting diagram:

$$\begin{array}{ccccccccc}
(*)\ 0 & \to & K & \to & \mathcal{O}_T^n & \to & Q & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
(*')\ 0 & \to & K' & \to & \mathcal{O}_T^n & \to & Q' & \to & 0
\end{array}$$

where the vertical maps are isomorphisms.

Definition: Let $F_{m,n} \in Sch_k/Sets$ be the functor defined by:

$$F_{m,n}(T) := \{\text{equivalence classes of exact sequences } (*)\} \text{ with } F_{m,n}(f) = f^*$$

Theorem: The Grassmannian $G(m,n)$ represents the functor $F_{m,n}$.

Proof: First, we define a transformation of functors: $u : F_{m,n} \to h_{G(m,n)}$ to the functor of points of $G(m,n)$ which we'll then invert.

Given a sequence $(*)$ on T, choose an affine covering $T = \cup \mathrm{Spec}(A_i)$ such that K may be trivialized over each $\mathrm{Spec}(A_i)$. Then over each such affine, the map $K|_{\mathrm{Spec(A_i)}} \cong \mathcal{O}^m_{\mathrm{Spec(A_i)}} \to \mathcal{O}^n_{\mathrm{Spec(A_i)}}$ from $(*)$ is given by an $m \times n$ array of elements of A_i, determining a morphism $f_i : \mathrm{Spec}(A_i) \to \mathbf{A}_k^{mn}$. The fact that the cokernel of $K \to \mathcal{O}_T^n$ is also locally free implies that each f_i factors through U. These morphisms do not patch together, but the **compositions** $\phi \circ f_i$ (for the map $\phi : U \to G(m,n)$ defined earlier) do patch to a morphism $f : T \to G(m,n)$, and this morphism does not depend upon the trivializations or the sequence $(*)$ within its equivalence class.

Thus $u(T) : F_{m,n}(T) \to h_{G(m,n)}(T)$ is defined on the level of objects. One easily checks that u is a transformation of functors.

To invert u, we find the universal family on $G(m,n)$. The key is to construct the sub-bundle $K \hookrightarrow \mathcal{O}^n_{G(m,n)}$. To do this, we use the action μ again. Recall the open cover $W_J := G(m,n) \cap U_{x_J} \cong Z_J \cong \mathbf{A}_k^{m(n-m)}$ of the Grassmannian from (iii) of the proof of the proposition. Putting this together with (i) of the same proof, we obtain isomorphisms:

$$\rho_{I,J} : \mathrm{GL}(m,k) \times (W_I \cap W_J) \cong \phi^{-1}(W_I \cap W_J) \cong \mathrm{GL}(m,k) \times (W_I \cap W_J)$$

identifying $W_I \cap W_J$ with open subsets of Z_I on the left side and Z_J on the right. The morphism to $\mathrm{GL}(m,k)$ determined by $\rho_{I,J}$ descends to a morphism $g_{I,J} : W_I \cap W_J \to \mathrm{GL}(m,k)$ which satisfies the cocycle condition, defining the locally free sheaf K of rank m on $G(m,n)$.

The embeddings $W_J \cong Z_J \hookrightarrow \mathbf{A}_k^{mn}$ give maps $\mathcal{O}_{W_J}^m \to \mathcal{O}_{W_J}^n$ which glue together via the transition functions to an injective map $K \hookrightarrow \mathcal{O}_{G(m,n)}^n$. The cokernel is determined up to isomorphism, and is locally free. This is our universal family. Via the pull-back applied to this family, we obtain a transformation $v : h_{G(m,n)} \to F_{m,n}$.

It is immediate from the constructions that u and v are inverses(!)

Corollary: $G(m,n) \cong G(n-m,n)$.

Proof: We only need to find an isomorphism of functors $F_{m,n} \cong F_{n-m,n}$ by the Theorem. But such an isomorphism is obtained by dualizing the exact sequences $(*)$ and fixing an isomorphism $k^n \cong (k^n)^*$, which then gives simultaneous isomorphisms $\mathcal{O}_T^n \cong (\mathcal{O}_T^n)^*$ for all the schemes T, hence the desired bijection on equivalence classes of exact sequences.

Note: Of course this isomorphism coincides with the identification of a subspace of k^n of dimension m with a subspace of $(k^n)^*$ of dimension $n-m$ by taking the dual of the cokernel. But without the Theorem we cannot deduce an isomorphsm of **schemes** from this bijection of k-rational points.

Variations: The proof goes through without modification if we replace "k" by "A" (any commutative ring with 1) giving us Grassmannians over $\mathrm{Spec}(A)$. If S is any scheme with affine open cover $S = \cup \mathrm{Spec}(A_i)$, then the Grassmannians over $\mathrm{Spec}(A_i)$ patch together to give us a Grassmannian $G(m,n)$ over S. But we can do even better than this. If $\mathcal{E}$ is any locally free sheaf of rank n on S with trivializations over the $\mathrm{Spec}(A_i)$, then the Grassmannians over the affines $\mathrm{Spec}(A_i)$ patch together to give a "twisted" Grassmannian $G(m,\mathcal{E})$ representing the functor from schemes over S to sets:

$$F_{m,\mathcal{E}}(T) = \{\text{equivalence classes of sequences } (*)\}$$

where $(*) : 0 \to K \to \mathcal{E}_T \to Q \to 0$ are short exact sequences of locally free sheaves on T with $\mathrm{rk}(K) = m$ and $\mathcal{E}_T$ is the pull-back of $\mathcal{E}$ under $T \to S$.

The fibers of $G(m,\mathcal{E})$ are just the Grassmannians $G(m,n)$ over $k(s)$ and the corollary generalizes to an isomorphism $G(m,\mathcal{E}) \cong G(n-m,\mathcal{E}^*)$.

Fix a Noetherian scheme S, a projective S-scheme X, a coherent sheaf $\mathcal{E}$ on X, and a polynomial $P(d)$. Given a Noetherian S-scheme T, let $\mathcal{E}_T$ be the pull-back of $\mathcal{E}$ to $X \times_S T$. For coherent sheaf quotients $(*) : \mathcal{E}_T \to \mathcal{Q} \to 0$ on $X \times_S T$, we declare that $(*) \sim (*')$ if there is a commuting diagram:

$$\begin{array}{cccccc} (*) & \mathcal{E}_T & \to & \mathcal{Q} & \to & 0 \\ & \| & & \downarrow & & \\ (*') & \mathcal{E}_T & \to & \mathcal{Q}' & \to & 0 \end{array}$$

We now define a functor from (Noetherian) S-schemes to sets:

$F_{\mathcal{E},P(d)}(T) = \{$equivalence classes of quotients $(*)$ such that $\mathcal{Q}$ is flat over T and each $\mathcal{Q}_t$ has Hilbert polynomial $P(d)\}$

$F_{\mathcal{E},P(d)}(f) = \widetilde{f}^*$ where $\widetilde{f}$ is the "base extension"

$$\begin{array}{ccc} X \times_S U & \stackrel{\widetilde{f}}{\to} & X \times_S T \\ \downarrow & & \downarrow \\ U & \stackrel{f}{\to} & T \end{array}$$

given a morphism $f : U \to T$ of S-schemes.

Example! If $X = S$, $P(d) = n - m$ and $\mathcal{E}$ is locally free of rank n, then $F_{\mathcal{E},P(d)}$ is the functor $F_{m,\mathcal{E}}$ which was represented by the Grassmannian! After all, $\mathcal{Q}$ is flat over $T = X \times_S T$ with Hilbert polynomial $n - m$ if and only if $\mathcal{Q}$ is locally free of rank $n - m$(!)

Theorem (Grothendieck's Quot Scheme): The functor $F_{\mathcal{E},P(d)}$ is always represented by a projective scheme, which we denote by $\mathrm{Quot}(\mathcal{E}, P(d))$.

Proof: As in the earlier variations on the Grassmannian, we may as well assume $S = \mathrm{Spec}(A)$, since the quot schemes will patch. We prove first:

A Simple Case: Suppose $X = \mathbf{P}^m_A$ and $\mathcal{E} = \mathcal{O}^n_{\mathbf{P}^m_A}(l)$ for some l, n.

Let $\mathcal{K}$ be the kernel of $(*) : \mathcal{E}_T \to \mathcal{Q}$. Then $\mathcal{E}_T = \mathcal{O}^n_{\mathbf{P}^m_T}(l)$ is evidently flat over T, hence $\mathcal{K}$ is flat over T as well, with constant Hilbert polynomial $P'(d) := n\binom{m+l+d}{l+d} - P(d)$ on the fibers. Since the $\mathcal{K}_t \hookrightarrow \mathcal{O}^n_{\mathbf{P}^n_t}(l)$ are each coherent subsheaves of a fixed locally free sheaf for each t, uniform vanishing applies. Moreover, it follows from the proof of uniform vanishing in this special case that d_0 only depends upon $P(d), m, n$ and l and not upon the field k, hence it yields the following "more uniform" result:

There is an d_0 independent of $(*)$ and t such that:

$$\mathrm{H}^i(\mathbf{P}^m_t, \mathcal{K}_t(d)) = 0 \text{ for all } t \in T, i > 0, d \geq d_0$$

Remark: This holds for finite fields as well. If $k(t) = \mathbf{F}_q$, then vanishing under the flat base change $\mathrm{Spec}(\overline{\mathbf{F}}_q) \to \mathrm{Spec}(\mathbf{F}_q)$ to the (infinite) algebraic closure implies vanishing for $\mathbf{F}_q$.

By flatness, it follows that the sequences: $0 \to \mathcal{K}_t \to \mathcal{E}_t \to \mathcal{Q}_t \to 0$ are all exact, and uniform vanishing for $\mathcal{K}_t$ and $\mathcal{E}_t = \mathcal{O}^n_{\mathbf{P}^m_t}(l)$ implies uniform vanishing for $\mathcal{Q}_t$. It follows from cohomology and base change that:

$$0 \to \pi_*\mathcal{K}(d_0) \to \pi_*\mathcal{E}_T(d_0) \to \pi_*\mathcal{Q}(d_0) \to 0$$

are exact, and moreover that $\pi_*\mathcal{K}(d_0)$ and $\pi_*\mathcal{Q}(d_0)$ are locally free sheaves of ranks $P'(d_0)$ and $P(d_0)$ respectively, where $\pi : \mathbf{P}^n_T \to T$ is the projection. Finally, note that $\pi_*\mathcal{E}_T(d_0)$ is a trivial vector bundle, naturally isomorphic to the pull-back of $\mathrm{H}^0(\mathbf{P}^m_A, \mathcal{O}^n_{\mathbf{P}^m_A}(l + d_0))$ from $\mathrm{Spec}(A)$.

Hence $(*)$ determines a morphism to the Grassmannian over $\mathrm{Spec}(A)$:

$$T \to G = G(P'(d_0), \mathrm{H}^0(\mathbf{P}^m_A, \mathcal{O}^n_{\mathbf{P}^m_A}(l + d_0)))$$

That is, we have defined $F_{\mathcal{E},P(d)} \to h_G$ on the level of objects. I leave it to you to use flatness again to prove that this is a transformation of functors. Our next task is to find the "image."

Consider the cokernel:

$$\pi^*K \to \mathcal{E}_G(d_0) \to \mathcal{F} \to 0$$

where the first map is the composition $\pi^*K \to \pi^*\pi_*\mathcal{E}_G(d_0) \to \mathcal{E}_G(d_0)$ and K is the universal subbundle on the Grassmannian. Let $\mathrm{Quot}(\mathcal{E}, P(d)) \subset G$ be the term in the flattening stratification of G with respect to $\mathcal{F}$ over which $\mathcal{F}$ is flat with Hilbert polynomial $P(d_0 + d)$. I claim that the transformation of functors factors through $\mathrm{Quot}(\mathcal{E}, P(d))$ and that the "universal quotient": $(*)$ $\mathcal{E}_{\mathrm{Quot}(\mathcal{E},P(d))} \to \mathcal{F}(-d_0)|_{\mathbf{P}^m_{\mathrm{Quot}}}$ inverts the transformation, proving that this "Quot" scheme represents the functor $F_{\mathcal{E},P(d)}$. To see this, note that uniform vanishing implies that $\mathcal{K}_t(d_0)$ is generated by its global sections for each t, and by cohomology and base change again, it follows that the natural map: $\pi^*\pi_*\mathcal{K}(d_0) \to \mathcal{K}(d_0)$ is surjective.

Thus if $T \to G$ is a morphism associated to a quotient $(*)$, then the cokernel above pulls back to:

$$\pi^*\pi_*\mathcal{K}(d_0) \to \mathcal{E}_T(d_0) \to \mathcal{Q}(d_0) \to 0$$

where the first map factors through $\mathcal{K}(d_0)$. This proves not only that the transformation factors through h_{Quot} but it also proves that the universal $(*)$ inverts this transformation, as desired.

So we have $\mathrm{Quot}(\mathcal{E}, P(d))$, given to us as a quasi-projective subscheme of a Grassmannian. To prove it is projective, the **valuative criterion** tells us that it suffices to prove that every morphism $\mathrm{Spec}(K) \to \mathrm{Quot} \to S$ from the fraction field of a discrete valuation ring R that extends to a morphism $\mathrm{Spec}(R) \to S$ also extends to a morphism $\mathrm{Spec}(R) \to \mathrm{Quot}$. In terms of the functor, this says that any flat quotient $(*)\mathcal{E}_K \to \mathcal{Q}$ on $\mathbf{P}^n_K$ extends uniquely to a flat quotient $(*)\mathcal{E}_R \to \overline{\mathcal{Q}}$ on $\mathbf{P}^n_R$. But this is exactly what the properness theorem (or rather its proof) gives us.

This completes the simple case.

Given a coherent sheaf $\mathcal{E}$ on a projective scheme $i : X \subset \mathbf{P}^m_A$, we may use Serre's theorem A to find a surjection $\mathcal{O}^n_X(l) \to \mathcal{E}$ giving rise to quotients: $\mathcal{O}^n_{\mathbf{P}^m_T}(l) \to i_{T*}\mathcal{E}_T$ for each S-scheme T, where $i_T : X_T \hookrightarrow \mathbf{P}^m_T$ is the induced closed immersion. In this way, flat quotients of $\mathcal{E}_T$ push forward under i_T to flat quotients of $\mathcal{O}^n_{\mathbf{P}^m_T}(l)$ and each $(*) : \mathcal{E}_T \to \mathcal{Q}$ determines a morphism $T \to \mathrm{Quot}(\mathcal{O}^n_{\mathbf{P}^m_A}(l), P(d))$ to the Quot scheme already constructed in the simple case. Let $\mathcal{G}$ be the kernel on $\mathbf{P}^m_{\mathrm{Quot}}$:

$$0 \to \mathcal{G} \to \mathcal{O}^n_{\mathbf{P}^m_{\mathrm{Quot}}}(l) \to i_{\mathrm{Quot}*}\mathcal{E}_{\mathrm{Quot}} \to 0$$

and choose $d_1 \geq d_0$ so that $\pi^*\pi_*\mathcal{G}(d_1) \to \mathcal{G}(d_1)$ is surjective. We have already seen that $\pi_*\mathcal{Q}(d_1)$ is locally free on Quot by cohomology and base change. It follows that the zero locus of $\pi_*\mathcal{G}(d_1) \to \pi_*\mathcal{Q}(d_1)$ is a closed subscheme $Z \subset \mathrm{Quot}(\mathcal{O}^n_{\mathbf{P}^m_A}(l), P(d))$. Over this subscheme, the universal quotient lifts to a quotient: $i_{Z*}\mathcal{E}_Z \to \mathcal{Q}|_{\mathbf{P}^m_Z}$ and it follows that $\mathcal{Q}|_{\mathbf{P}^m_Z}$ is the push-forward of a sheaf on X_Z which is the universal quotient for $\mathcal{E}_Z$. In other words, $\mathrm{Quot}(\mathcal{E}, P(d)) := Z$ represents $F_{\mathcal{E},P(d)}$.

Definition (The Hilbert Scheme): The Hilbert scheme $H_{X,P(d)}$ is the special case of the quot scheme representing the functor $F_{\mathcal{O}_X,P(d)}$.

Alternatively, the Hilbert scheme represents the functor:

$$F_{X,P(d)}(T) = \{\text{closed subschemes } Z \subset X \times_S T \text{ that are flat over } T \\ \text{with Hilbert polynomial } P(d)\}$$

via the identification of $i : Z \hookrightarrow X_T$ with the quotient $\mathcal{O}_{X_T} \to i_*\mathcal{O}_Z$.

Example (Mori): Suppose X is a projective scheme over Spec($\mathbf{Z}$) and $X_{\mathbf{F}_p}$ has a rational curve C (i.e. $\chi(C, \mathcal{O}_C) = 1$) of degree d for infinitely many primes p and fixed d. Then $X_{\mathbf{Q}}$ has a rational curve of degree d, as well. Indeed, the Hilbert polynomial of C is $P(d) = d + 1$, and since the Hilbert scheme $H_{X,d+1}$ is projective over Spec($\mathbf{Z}$), it must have only finitely many components. Thus it has a component whose image in Spec($\mathbf{Z}$) contains infinitely many primes. Such a component must dominate Spec($\mathbf{Z}$), and must therefore surject onto Spec($\mathbf{Z}$) by properness. A point in the fiber of such a component over Spec($\mathbf{Q}$) yields the desired rational curve.

Observation: Quot schemes "commute with base chage". That is, if $\mathcal{F}$ is a coherent sheaf on a T-scheme Y and we are given a morphism $S \to T$, then the Quot schemes fit into a fiber square:

$$\begin{array}{ccc} \mathrm{Quot}(\mathcal{F}_S, P(d)) & \to & \mathrm{Quot}(\mathcal{F}, P(d)) \\ \downarrow & & \downarrow \\ S & \to & T \end{array}$$

This follows from the theorem since the fiber product $\mathrm{Quot}(\mathcal{F}, P(d)) \times_T S$ represents the functor $F_{\mathcal{F}_S,P(d)}$.

Example (Maps to the Grassmannian): Suppose X is a projective k-scheme for some field k and suppose $P(d)$ is the Hilbert polynomial of a locally free sheaf Q of rank $n - m$ which is a quotient of a trivial bundle via $(*) : \mathcal{O}_X^n \to Q$. Then $(*)$ determines a k-rational point $q \in \mathrm{Quot}(\mathcal{O}_X^n, P(d))$ and also a morphism $f_q : X \to G(m, n)$. Moreover, the universal quotient $\mathcal{O}^n_{X\times\mathrm{Quot}} \to \mathcal{Q}$ is locally free in a neighborhood of $X \times \{q\}$ and it follows from properness that there is an open neighborhood $q \in U$ in the Quot scheme such that $\mathcal{Q}|_{X\times U}$ is locally free of rank $n - m$, hence all the points in U parametrize morphisms $X \to G(m, n)$. Thus we may think of the component(s) of the quot scheme containing U as a compactification of a space of maps from X to the Grassmannian.

Example (Another Look at the Grassmannian): Suppose $P(d)$ is the Hilbert polynomial of a linear subspace $V \subset \mathbf{P}_k^{n-1}$ of (projective) dimension $m-1$. Then $H_{\mathbf{P}_k^{n-1},P(d)} \cong G(m,n)$. Indeed, out of the universal subbundle $K \hookrightarrow \mathcal{O}_G^n$, one obtains an inclusion of projective bundles $P(K) \subset \mathbf{P}_k^{n-1} \times G$ which is a flat family over G. This gives a morphism $G(m,n) \to H_{\mathbf{P}_k^{n-1},P(d)}$. On the other hand, one can show that the only projective subschemes of $\mathbf{P}_k^{n-1}$ with Hilbert polynomial $P(d)$ are the linear subspaces, and that this morphism is an isomorphism.

Last Example (Hypersurfaces in Projective Space): The complete linear series $\mathbf{P}_k^{\binom{n+d}{n}-1}$ of hypersurfaces of degree d in $\mathbf{P}_d^n$ is a Hilbert scheme. Indeed, let $x_0, ..., x_n$ be the coordinates on $\mathbf{P}_k^n$ and let a_J for $J = (j_0, j_1, ..., j_n)$ satisfying $\sum j_i = d$ be the coordinates on $\mathbf{P}_k^{\binom{n+d}{n}-1}$. Then:

$$V(\sum_J a_J x_0^{j_0} x_1^{j_1} ... x_n^{j_n}) \subset \mathbf{P}_k^n \times \mathbf{P}_k^{\binom{n+d}{n}-1}$$

is the universal divisor, exhibiting $\mathbf{P}_k^{\binom{n+d}{n}-1}$ as the Hilbert scheme for hypersurfaces of degree d. Again, one checks that the only projective subscheme of $\mathbf{P}_k^n$ with the Hilbert polynomial of a hypersurface is in fact a hypersurface.