2.3 Roots

Roots are the key to a deeper understanding of polynomials.

Definition: Any value $r \in F$ that solves:

$$f(r) = 0$$

is called a **root** of the polynomial $f(x) \in F[x]$.

Examples: (a) Every $f(x) \in \mathbb{R}[x]$ of odd degree has at least one real root. The graph of $y = f(x)$ crosses $y = 0$ at least once (Intermediate Value Theorem) and if r is the x-coordinate of a crossing point, then $f(r) = 0$. In other words, each crossing point produces a root.

(b) Linear polynomials always have one root. We can be specific in this case. The linear polynomial $f(x) = ax + b$ has $r = -\frac{b}{a}$ as its one and only root.

Proposition 2.3.1. *(a) If $x - r$ divides $f(x)$, then r is a root of $f(x)$.*

(b) Conversely, if $x - r$ doesn't divide $f(x)$, then r isn't a root of $f(x)$.

Proof: If $x - r$ divides $f(x)$ then $f(x) = (x - r)q(x)$ and so:

$$f(r) = (r - r)q(r) = 0 \cdot q(r) = 0$$

This gives (a). If $x - r$ doesn't divide $f(x)$, division with remainders gives a **constant** remainder: $f(x) = (x - r)q(x) + a$ so that

$$f(r) = (r - r)q(r) + a = a \neq 0$$

This gives (b).

Corollary 2.3.2. *A polynomial of degree d has at most d different roots.*

Proof: Let $\{r_1, ..., r_n\}$ be any set of distinct roots of $f(x)$. We need to prove that $n \leq d$. Since r_1 is a root, $f(x) = (x - r_1)q_1(x)$ by the Proposition. All the other roots must also be roots of $q_1(x)$, since $f(r_i) = (r_i - r_1)q_1(r_i) = 0$ and $r_i - r_1 \neq 0$. In particular, $q_1(x) = (x - r_2)q_2(x)$, and we can continue the process, getting a string of equalities:

$$f(x) = (x - r_1)q_1(x) = (x - r_1)(x - r_2)q_2(x) = \cdots$$

$$= (x - r_1) \cdots (x - r_n)q_n(x)$$

Thus $n \leq d$ because $d = \deg(f(x)) = n + \deg(q_n(x))$.

Of course a polynomial of degree d could have **fewer** than d roots. A prime polynomial of degree ≥ 2, for example, has **no** roots at all.

Proposition 2.3.3 (The Rational Roots Test). *The only possible rational roots of a polynomial with* **integer** *coefficients:*

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + ... + a_0$$

are the rational numbers a/b (written in lowest terms) such that b divides a_d and a divides a_0

Proof: Suppose a/b is a rational root in lowest terms. Then

$$a_d\left(\frac{a}{b}\right)^d + a_{d-1}\left(\frac{a}{b}\right)^{d-1} + ... + a_1\left(\frac{a}{b}\right) + a_0 = 0.$$

If we clear denominators by multiplying through by b^d, we get:

$$a_d(a^d) + a_{d-1}(a^{d-1}b) + ... + a_1(ab^{d-1}) + a_0(b^d) = 0$$

and we can put $a_0(b^d)$ to one side of the equation and collect an a out of each of the terms on the other side to get:

$$a_0(b^d) = a\left(-a_d a^{d-1} - a_{d-1}a^{d-2}b - ... - a_1 b^{d-1}\right)$$

and so we see that a divides $a_0(b^d)$. Similarly:

$$a_d a^d = b\left(-a_{d-1}a^{d-1} - a_{d-2}a^{d-2}b - ... - a_0 b^{d-1}\right)$$

so we see that b divides $a_d(a^d)$. But we chose a/b to be in lowest terms, so none of the prime factors of a and of b are the same. It follows that a divides a_0 and b divides a_d.

This gives us the following:

Strategy for finding all rational roots of $f(x)$ with integer coefficients:

Step 1: Assemble all a/b's for which a divides a_0 and b divides a_d.

Step 2: The ones that solve $f(a/b) = 0$ are all the rational roots.

Example: Find the rational roots of $2x^3 + 11x^2 + 17x + 6$. First, assemble:

$$\frac{1}{2}, -\frac{1}{2}, 1, -1, \frac{3}{2}, -\frac{3}{2}, 2, -2, 3, -3, 6, -6$$

and then try them all!

$$\begin{array}{lllllllllllll}
f(\frac{1}{2}) & = & \frac{35}{2} & f(-\frac{1}{2}) & = & 0 & f(1) & = & 36 & f(-1) & = & -2 \\
f(\frac{3}{2}) & = & 63 & f(-\frac{3}{2}) & = & -\frac{3}{2} & f(2) & = & 100 & f(-2) & = & 0 \\
f(3) & = & 210 & f(-3) & = & 0 & f(6) & = & 936 & f(-6) & = & -132
\end{array}$$

Thus $-1/2, -2$ and -3 are the rational roots.

Corollary 2.3.4. **None** *of the* n*th roots* $\sqrt[n]{2}$ *is rational (for* $n > 1$*).*

Proof: An nth root of 2 is, by definition, a root of the polynomial:

$$f(x) = x^n - 2$$

But the only possible rational roots of $f(x)$ are $1, -1, 2, -2$ by the test. Since none of these solve $x^n - 2 = 0$, we see that $\sqrt[n]{2}$ isn't rational!

Amusing Observation: This is our third proof that $\sqrt{2}$ is irrational. This one, however, proves much more.

Definition: Any complex number that is a root of a polynomial in $\mathbb{Q}[x]$ is called an **algebraic number** (or just **algebraic**).

Proposition 2.3.5. *If* $\alpha = s + it$ *is an algebraic number, then* $\overline{\alpha} = s - it$ *is an algebraic number, too.*

Proof: If α is algebraic, then by definition, $f(\alpha) = 0$ where

$$f(x) = a_d x^d + ... + a_0 \text{ and each } a_i \text{ is rational.}$$

But then $0 = \overline{0} = \overline{f(\alpha)} = \overline{a}_d \overline{\alpha}^d + ... + \overline{a}_0 = f(\overline{\alpha})$ because complex conjugation is linear and multiplicative! So $\overline{\alpha}$ is a root of the **same** polynomial $f(x)$.

Proposition 2.3.6. *If* $f(x)$ *and* $g(x) \in \mathbb{Q}[x]$ *have a* **complex** *root in common, then "the" gcd of* $f(x)$ *and* $g(x)$ *has* **positive** *degree.*

Proof: Euclid's algorithm **gives the same result** for the gcd of $f(x)$ and $g(x)$ whether we think of them as polynomials in $\mathbb{Q}[x]$ or as polynomials in $\mathbb{C}[x]$ (See Exercise 5-3). As polynomials in $\mathbb{C}[x]$, they have a common factor, namely $x - \alpha$, where α is the complex root they have in common (Proposition 2.3.1). So the gcd, whether thought of in $\mathbb{C}[x]$ or in $\mathbb{Q}[x]$, has positive degree.

Example: $x^4 + 2x^2 + 1$ and $x^4 + 3x^2 + 2$ have no rational roots at all. They do have the complex root i in common, and $x^2 + 1$ is a gcd.

Proposition 2.3.7. *If* $\alpha = s + it$ *is an algebraic number, there is* **exactly one** *prime polynomial* $p(x) \in \mathbb{Q}[x]$ *with* α *as a root and of the form:*

$$p(x) = x^d + a_{d-1}x^{d-1} + ... + a_0$$

Proof: By definition, α is a root of **some** polynomial $f(x) \in \mathbb{Q}[x]$. If we factorize: $f(x) = p_1(x) \cdots p_n(x)$ in $\mathbb{Q}[x]$, then $0 = f(\alpha) = p_1(\alpha)p_2(\alpha) \cdots p_n(\alpha)$ so (at least) one of the $p_i(\alpha) = 0$. Thus α is a root of **some** prime polynomial. Suppose there are two prime polynomials $p(x)$ and $q(x)$ with α as a root. By Proposition 2.3.6, we know that their gcd has positive degree. But the gcd must have the same degree as $p(x)$ and as $q(x)$ since they are prime, and it follows that $p(x)$ is a constant (rational) multiple of $q(x)$. There is exactly one constant multiple of any polynomial that has the form $x^d + a_{d-1}x^{d-1} + \cdots a_0$, so there is exactly one such prime polynomial with α as a root.

Definition: Given an algebraic number α, the prime polynomial $p(x)$ of the Proposition is called the **characteristic polynomial** of α.

Remark: The algebraic number α "knows" its characteristic polynomial $p(x)$, so it also knows all the **other** roots of $p(x)$. These other roots are called the **(algebraic) conjugates** of α.

Examples: (a) The golden mean $\frac{-1+\sqrt{5}}{2}$ has characteristic polynomial:

$$x^2 + x - 1$$

and its algebraic conjugate is "little" golden mean: $\frac{-1-\sqrt{5}}{2}$.

(b) The characteristic polynomial of $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ is:

$$x^4 + 1$$

and there are three algebraic conjugates, which are the other fourth roots of -1:

$$-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \quad \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \text{ and } -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

Notice that the "ordinary" complex conjugate is one of the algebraic conjugates. This is always true of algebraic numbers that are not real (see Proposition 2.3.5).

A Hard Question: Which complex numbers are algebraic numbers?

The classical formulas for the roots of low degree polynomials give some clues.

The Quadratic Formula: The roots of $ax^2 + bx + c \in \mathbb{Q}[x]$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $\pm\sqrt{b^2 - 4ac}$ are the square roots of $b^2 - 4ac$.

Proof: Divide through by a and complete the square:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x + \frac{b}{2a})^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right) = 0$$

The solutions are then:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition: $\Delta = b^2 - 4ac$ is the **discriminant** of $ax^2 + bx + c$.

Corollary 2.3.8. *If $ax^2 + bx + c \in \mathbb{Q}[x]$ then:*

(i) if $\Delta > 0$, there are two roots, both real.

(ii) if $\Delta = 0$, the two roots come together to one real root.

(iii) if $\Delta < 0$, there are two roots, both complex (i.e. not real).

The Cubic Formula: The roots of

$$f(x) = ax^3 + bx^2 + cx + d \in \mathbb{Q}[x]$$

may be obtained as follows:

Preliminary Step: Divide through by a.

Next Step: Complete the **cube**. This is already a little messy:

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = (x + \frac{b}{3a})^3 + (\frac{c}{a} - \frac{b^2}{3a^2})(x + \frac{b}{3a}) + (\frac{d}{a} - \frac{bc}{3a^2} + \frac{2b^3}{27a^3}) = 0$$

We change variables before proceeding:

$$y = x + \frac{b}{3a}, \quad p = \frac{c}{a} - \frac{b^2}{3a^2} \quad \text{and} \quad q = \frac{d}{a} - \frac{bc}{3a^2} + \frac{2b^3}{27a^3}$$

and then the roots of $y^3 + py + q = 0$ minus $b/3a$ are the roots of $f(x)$.

If we are really lucky and $q = 0$, then the roots are $y = 0, \pm\sqrt{-p}$ so:

$$x = -\frac{b}{3a} \text{ and } x = -\frac{b}{3a} \pm \sqrt{\frac{b^2}{3a^2} - \frac{c}{a}} = \frac{-b \pm \sqrt{3b^2 - 9ac}}{3a}$$

are the roots of $f(x)$. This looks a bit like the quadratic formula, which is no accident. The first root could have been found with the rational roots test, and then the other two by the quadratic formula.

If we are only a little lucky and $p = 0$, then the roots are $y = \sqrt[3]{-q}$ for the **three** complex cube roots of $-q$, and then the roots of $f(x)$ are:

$$x = \frac{-b}{3a} + \sqrt[3]{\frac{bc}{3a^2} - \frac{2b^3}{27a^3} - \frac{d}{a}} = \frac{-b + \sqrt[3]{b^3 - 27a^2d}}{3a}$$

which also looks a bit like the quadratic formula (but it produces three roots)! Otherwise $p \neq 0$ and $q \neq 0$, and we turn to the:

Second Step (an inspired guess): Set

$$y = z - \frac{p}{3z}$$

Then:

$$y^3 + py + q = (z - \frac{p}{3z})^3 + p(z - \frac{p}{3z}) + q = z^3 - \frac{p^3}{27z^3} + q$$

which we multiply through by z^3 to get a degree 6 equation:

$$z^6 + qz^3 - \frac{p^3}{27} = 0$$

This looks like a strange thing to do, since it created a degree 6 polynomial. However, we can use the quadratic formula to find the solutions to this:

$$z^3 = \frac{-q + \sqrt{q^2 + \frac{4p^3}{27}}}{2} = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

so that the roots of $z^6 + qz^3 - \frac{p^3}{27}$ are:

$$z = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

and then the roots of $y^3 + py + q$ are:

$$y = z - \frac{p}{3z} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}$$

and finally, the roots of $f(x)$ are:

$$x = -\frac{b}{3a} + y = -\frac{b}{3a} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}$$

This looks like it gives six numbers (one for each square root and cube root). But it actually only produces 3 different numbers.

Two "Easy" Examples (already of the form $y^3 + py + q$).

(a) Find the roots of $y^3 - y + 1$. Here $p = -1, q = 1$ and so:

$$y = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{27}}} + \frac{1}{3\sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{27}}}}$$

From the positive square root: $\sqrt{\frac{1}{4} - \frac{1}{27}} \approx 0.46148$ we get:

$$z = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{27}}} \approx \sqrt[3]{-0.03852} \approx 0.3377\sqrt[3]{-1}$$

In polar coordinates:

$$z \approx (0.3377; \frac{\pi}{3}), (0.3377; \pi), (0.3377; \frac{5\pi}{3})$$

When we plug into the formula for y we get the roots:

$$y \approx .6624 - 0.5624i,\ x \approx -1.325,\ \text{and}\ y \approx .6624 + 0.5624i$$

(b) Find the roots of $y^3 - 2y + 1$. Here $p = -2, q = 1$ and so:

$$y = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{8}{27}}} + \frac{2}{3\sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{8}{27}}}}$$

Right away we get complex numbers, since $\frac{1}{4} - \frac{8}{27}$ is negative.

$$z = \sqrt[3]{-\frac{1}{2} + i\sqrt{\frac{8}{27} - \frac{1}{4}}} \approx \sqrt[3]{(0.54433; 2.73521)}$$

This looks nasty! The cube roots are (approximately):

$$(0.8165; 0.9118), (0.8165; 3.0061) \text{ and } (0.8165; 5.1005)$$

and when we plug these in to the formula for y, we get a surprise!

$$y \approx 0.9999, -1.6180, 0.6180$$

which are familiar **real** numbers. The first is 1 (within margin of error) and the others are (also within margin of error) the golden mean and its conjugate! Looking back at the polynomial, we notice that it factors.

$$y^3 - 2y + 1 = (y - 1)(y^2 + y - 1)$$

so this shouldn't have been a surprise after all.

Note: In the first example, only one root was real, while in the second there were three real roots. There is a discriminant to detect this:

Definition: The **discriminant** of $ax^3 + bx^2 + cx + d$ is

$$\Delta = a^4 3^3 2^2 \left(\frac{q^2}{4} + \frac{p^3}{27}\right)$$

and this gets pretty complicated when we substitute for a, b, c, d:

$$\Delta = 27a^2d^2 - 18abcd + 4ac^3 + 4b^3d - b^2c^2$$

Corollary 2.3.9. *If $ax^3 + bx^2 + cx + d \in \mathbb{Q}[x]$, then:*

(i) if $\Delta > 0$, then there are three roots; one real and two complex.

(ii) if $\Delta = 0$, then two (or three) roots come together; all are real.

(iii) if $\Delta < 0$, then there are three roots; all real.

Note: Unlike the quadratic case, this isn't obvious at all. In fact, it is pretty counterintuitive. When $\Delta < 0$, the square root is purely imaginary, which means that in order to come up with three real roots, we are **forced** to go through the "nastiness" with the complex numbers!

Proof: Consider:

$$f(y) = y^3 + py + q$$

the polynomial we got after completing the cube in the cubic formula. Since the roots of the original polynomial are translates by $-\frac{b}{3a}$ of the roots of $f(y)$, we may work with $f(y)$ instead.

Also, forget about the factor $a^4 3^3 2^2$, since it is always a positive number. (This factor is only there to make discriminant look nicer!)

The idea is to study critical points of $f(y)$. These are the two solutions to:

$$f'(y) = 3y^2 + p = 0; \quad \text{namely} \quad y = \pm\sqrt{-\frac{p}{3}}$$

Two roots come together when a critical point is **also** a solution of $f(y) = 0$. (Exercise!) Substituting $y = \pm\sqrt{-\frac{p}{3}}$ into $y^3 + py + q = 0$, this gives us:

$$\left(\pm\sqrt{-\frac{p}{3}}\right)^3 + p\left(\pm\sqrt{-\frac{p}{3}}\right) + q = 0$$

so

$$q = \mp\frac{2}{3}\sqrt{-\frac{p}{3}} \quad \text{and then} \quad q^2 = -\frac{4p^3}{27}$$

(squaring both sides). But this is exactly what we get when we set $\Delta = 0$. It also tells us that $p \leq 0$ at each critical point (because $q^2 \geq 0$), and so the critical point is always a **real** root (either $\sqrt{-\frac{p}{3}}$ or $-\sqrt{-\frac{p}{3}}$). This only leaves one more root which has to be real, too, because if it weren't, its complex conjugate would be an additional root (see Proposition 2.3.5).

Next, notice that if $p \geq 0$ then $f(y)$ has zero (or one) real critical points, so it is a strictly increasing function, and therefore has only one **real** root! In fact, the only way $f(y)$ can have three real roots is if $p < 0$ and if the critical points $c_1 = -\sqrt{-\frac{p}{3}}$ and $c_2 = \sqrt{-\frac{p}{3}}$ satisfy $f(c_1) > 0$ and $f(c_2) < 0$. (Think about the graph of $f(y)$.)

Substituting into $f(y)$ (and remembering that $p < 0$), we see that:

$$f(c_1) = -\frac{2p}{3}\sqrt{-\frac{p}{3}} + q > 0 \quad \text{and} \quad f(c_2) = \frac{2p}{3}\sqrt{-\frac{p}{3}} + q < 0$$

both happen exactly when:

$$|q| < |\frac{2p}{3}|\sqrt{-\frac{p}{3}} \quad \text{and} \quad q^2 < -\frac{4p^3}{27}$$

which is to say, exactly when $\Delta < 0$. This the Corollary.

Examples: (a) $x^3 - 3x + 4$ has only one real root because $\Delta = 324$.

(b) $x^3 - 3x + 1$ has three real roots because $\Delta = -81$.

(c) $x^3 - 3x + 2$ has less than three roots because $\Delta = 0$.

Just to see that it is possible, here is (without proof):

The Quartic Formula: The four roots of:

$$ax^4 + bx^3 + cx^2 + dx + e \in \mathbb{Q}[x]$$

are obtained as follows:

First Step: Divide through by a and complete the **quartic**, to get:

$$y^4 + py^2 + qy + r = 0$$

with the substitutions:

$$y = x + \frac{b}{4a}$$

$$p = \frac{c}{a} - \frac{3b^2}{8a^2}$$

$$q = \frac{d}{a} - \frac{bc}{2a^2} + \frac{b^3}{8a^3}$$

$$r = \frac{e}{a} - \frac{bd}{4} + \frac{b^2c}{16a^3} - \frac{3b^4}{256a^4}$$

Second Step (a truly inspired guess): Solve a different cubic(!)

$$g(y) = y^3 - 2py^2 + (p^2 - 4r)y + q^2 = 0$$

with the cubic formula, and let s_1, s_2, s_3 be the roots. Then it turns out that:

$$-\frac{b}{4a} + \frac{\sqrt{-s_1} + \sqrt{-s_2} + \sqrt{-s_3}}{2}, \quad -\frac{b}{4a} + \frac{\sqrt{-s_1} - \sqrt{-s_2} - \sqrt{-s_3}}{2}$$

$$-\frac{b}{4a} - \frac{\sqrt{-s_1} + \sqrt{-s_2} - \sqrt{-s_3}}{2} \quad \text{and} \quad -\frac{b}{4a} - \frac{\sqrt{-s_1} - \sqrt{-s_2} + \sqrt{-s_3}}{2}$$

are the four roots of the original quartic!

Remark: This inspired guess is very misleading! It seems to suggest that trickery will allow you to solve high degree polynomials by solving lower degree ones and combining the roots in clever ways. This turns out to be impossible already for degree 5 polynomials, as we will see.

2.3.1 Roots Exercises

7-1 Define the **derivative transformation** on polynomials:

$$\frac{d}{dx} : F[x] \to F[x] \quad \text{by setting} \quad \frac{dx^n}{dx} = nx^{n-1}$$

By calling it a transformation, I am putting linearity into the definition:

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \quad \text{and} \quad \frac{d}{dx}(kf) = k\frac{df}{dx}$$

(a) Prove Leibniz' rule:

$$\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$$

Hint: You only need to prove it when $f(x) = x^n$ and $g(x) = x^m$.

(b) Prove that if $(x-r)^2$ divides $f(x)$, then r is a root of $f(x)$ **and** $\frac{df}{dx}(x)$.

(c) Prove the converse of (b) (see Proposition 2.3.1 (b)).

(d) If $f(x) \in \mathbb{Q}[x]$ is the characteristic polynomial of α, prove that $(x-\alpha)^2$ does not divide $f(x)$. (So an algebraic number is never a conjugate of itself.)

Hint: Think about the gcd of $f(x)$ and of $\frac{df}{dx}(x)$.

(e) If $f(x) \in \mathbb{Q}[x]$ has degree n, then prove that for each $x_0 \in \mathbb{Q}$:

$$f(x) = f(x_0) + \frac{1}{1!}\frac{df}{dx}(x_0)(x-x_0) + ... + \frac{1}{n!}\frac{d^n f}{dx^n}(x_0)(x-x_0)^n$$

In other words, each polynomial has a Taylor expansion at each x_0.

Hint: Again, you only need to check it for $f(x) = x^n$ (see Exercise 5-1).

7-2 Find **all** of the roots (some only approximate) of:

(a) $x^3 + x + 2$ (b) $x^3 + x - 3$

(c) $x^3 - 3x + 1$ (d) $x^3 - 3x + 2$

(e) $x^3 + x^2 + x + 1$ (f) $x^3 + x^2 + x + 2$

7-3 Find the characteristic polynomials and all conjugates of:

(a) $\frac{1-\sqrt{7}}{2}$ (b) $\frac{1+\sqrt{7}\,i}{2}$ (c) $\sqrt[3]{2}$

(d) $\frac{-1+\sqrt[3]{-26}}{3}$ (e) $\sqrt{2}+\sqrt{3}\,i$ (f) $\sqrt{\frac{3+\sqrt{2}}{2}}$