1.3 The Real Numbers.

The real numbers:

$$\mathbb{R} = \{\text{numbers on the number-line}\}$$

require some real analysis for a "proper" definition. We'll sidestep the analysis, relying instead on our less precise notions of continuity from calculus. Notice that the real numbers are ordered (from left to right) and come in three types:

$$\mathbb{R} = \mathbb{R}^- \cup \{0\} \cup \mathbb{R}^+$$

where

$$\mathbb{R}^+ = \{\text{positive real numbers}\}$$

are the real numbers that measure lengths (just as the natural numbers count). Notice also that rational numbers are examples of real numbers. We didn't define the rational numbers to be numbers on the number-line, but since the slope of a line through $(0,0)$ and (b,a) is the y-coordinate of its intersection with the vertical line $x = 1$, we may think about our number-line in that way (as the vertical line $x = 1$), and then

$$\mathbb{Q} \subset \mathbb{R} = \{\text{slopes of } \textbf{all} \text{ lines through } (0,0) \text{ (except the } y\text{-axis)}\}$$

We want to see that the real numbers are a field (see 1.1.2) and that "most" of the real numbers are not rational (remember $\sqrt{2}$). In fact, we will be able to find plenty of irrational numbers using:

Decimals: An **infinite decimal** is a sequence of the following form:

$$q.d_1d_2d_3\cdots$$

where q is a whole number (a natural number or zero), and each d_i is a digit (whole number between 0 and 9). All the decimals we will use will be infinite. A **terminating decimal** is a decimal

$$q.d_1d_2\cdots d_n$$

which we will make infinite by padding it with zeroes:

$$q.d_1d_2\cdots d_n00\cdots$$

A terminating decimal always represents a rational number:

$$q.d_1d_2\cdots d_n00\cdots = q + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

(Remember, we no longer use the brackets when writing rational numbers!)

A non-terminating decimal represents a real number in the same way, except that we need the notion of **convergence** from calculus to make sense of the infinite sum:

$$q + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots$$

Conversely, every (positive) real number has a decimal expansion.

Definition of Decimal Expansions: Given a positive real number $r \in \mathbb{R}^+$, the (infinite) decimal expansion of r is defined as follows:

q is chosen so that $q \leq r < q+1$. The digits are then chosen by induction:

(i) d_1 is chosen so that:

$$q + \frac{d_1}{10} \leq r < q + \frac{d_1}{10} + \frac{1}{10}$$

It is between 0 and 9 because $q \leq r < q+1$.

(ii) Each d_{n+1} is chosen so that:

$$q + \frac{d_1}{10} + \cdots + \frac{d_{n+1}}{10^{n+1}} \leq r < q + \frac{d_1}{10} + \cdots + \frac{d_{n+1}}{10^{n+1}} + \frac{1}{10^{n+1}}$$

It is between 0 and 9 because:

$$q + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n} \leq r < q + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n} + \frac{1}{10^n}$$

This gives a **sequence** of terminating decimals (= rational numbers):

$$q,\ q.d_1,\ q.d_1d_2,\ q.d_1d_2d_3,\ \text{etc.}$$

that converges to r. Thus, $r = q.d_1d_2d_3\cdots$.

Examples: (a) The natural number m expands as the terminating decimal:

$$m.000000\cdots$$

The infinite decimal:

$$(m-1).99999\cdots$$

also represents m, but you will never get it as the decimal expansion. All the terminating decimals (and only the terminating decimals) have this ambiguity.

(b) The decimal expansion of $1/3$ is $0.33333\cdots$ because

$$0.333\cdots 3 < \frac{1}{3} < 0.333\cdots 4$$

no matter how many digits we take (multiply through by 3 to see this).

(c) We can decimal expand $\sqrt{2}$ as far as we want by squaring:

$1^2 = 1 < 2 < 4 = 2^2$, so $1 < \sqrt{2} < 2$ so $q = 1$.

$(1.4)^2 = 1.96 < 2 < 2.25 = (1.5)^2$, so $d_1 = 4$.

$(1.41)^2 = 1.9881 < 2 < 2.0164 = (1.42)^2$, so $d_2 = 1$.

$(1.414)^2 = 1.999396 < 2 < 2.002225 = (1.415)^2$, so $d_3 = 4$.

Remark: You will frequently see: $\sqrt{2} = 1.414\cdots$. Unlike the decimal $0.333\cdots$ above, this use of "$\cdots$" means only that 1.414 are the first three digits of the infinite decimal expansion of $\sqrt{2}$. It does **not** mean that there is a pattern!

Expanding Rationals. Given a positive rational number $\frac{l}{m}$, perform the following divisions with remainders to define the digits of a decimal:

First, set $l = mq + r$ (this defines q and a whole number $r < m$)

Next, define the digits by induction:

(i) Set $10r = md_1 + r_1$ (this defines d_1, which is a digit, and $r_1 < m$)

(ii) Set each $10r_n = md_{n+1} + r_{n+1}$
(this defines d_{n+1}, which is a digit, as well as $r_{n+1} < m$)

and this defines digits d_n (and remainders $r_n < m$) for all n by induction.

This looks sort of like Euclid's algorithm, except this one never ends. But if $r_n = 0$ for some n, then $0 = d_{n+1} = d_{n+2} = \cdots$ and the decimal terminates.

You should convince yourself that this algorithm for expanding rationals is exactly how you were taught to find the decimal of a rational number as the "long divison" of l by m. But now we are in a position to prove that this expansion gives the **correct** decimal expansion of $\frac{l}{m}$!

Proposition 1.3.1. *The infinite decimal in the rational expansion of* $\frac{l}{m}$ *is equal to its decimal expansion.*

Proof: To get started, divide $l = mq + r$ by m to get:

$$(*) \quad \frac{l}{m} = q + \frac{r}{m} \quad \text{and then} \quad q \le \frac{l}{m} < q + 1 \text{ (because } 0 \le \frac{r}{m} < 1)$$

so this is the correct q. Next, a proof by induction checks the decimals:

(i) Divide $10r = md_1 + r_1$ by $10m$ to get $\frac{r}{m} = \frac{d_1}{10} + \frac{r_1}{10m}$, and substitute into $(*)$ to get:

$$\frac{l}{m} = q + \frac{d_1}{10} + \frac{r_1}{10m}$$

which proves that d_1 is the correct digit (because $0 \le \frac{r_1}{m} < 1$)!

(ii) Once we know that $d_1, ..., d_n$ are the correct first n digits, and in fact that:

$$(**) \quad \frac{l}{m} = q.d_1 d_2 \cdots d_n + \frac{r_n}{10^n m}$$

then divide $10r_n = md_{n+1} + r_{n+1}$ by $10^{n+1}m$ to get $\frac{r_n}{10^n m} = \frac{d_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1}m}$, and substitute into $(**)$ to get:

$$\frac{l}{m} = q.d_1 d_2 \cdots d_n + \frac{d_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1}m} = q.d_1 d_2 \cdots d_{n+1} + \frac{r_{n+1}}{10^{n+1}m}$$

This proves that d_{n+1} is also correct, and completes the proof by induction.

Definition: A **repeating decimal** is any decimal of the form:

$$q.d_1 d_2 \cdots d_k d_{k+1} \cdots d_n d_{k+1} \cdots d_n d_{k+1} \cdots d_n \cdots$$

for some pair of natural numbers $k < n$. We write this as:

$$q.d_1 d_2 \cdots d_k \overline{d_{k+1} \cdots d_n}$$

to avoid the "$\cdots$" ambiguity remarked upon earlier.

Example: Your calculator's output for 1/35 will convince you that:

$$\frac{1}{35} = 0.0\overline{285714} \quad (k = 1, n = 7)$$

We will prove this as we prove the following:

Proposition 1.3.2. *All the decimal expansions of rational numbers repeat.*

Proof: Consider again step (ii) in the rational expansion of l/m above:

$$(ii) \quad 10r_n = md_{n+1} + r_{n+1}$$

From this step, it follows that if $r_k = r_n$ for some $k < n$, then:

$$d_{k+1} = d_{n+1} \text{ and } r_{k+1} = r_{n+1}$$

because they are the quotients and remainders when the **same** numbers $10r_k = 10r_n$ are divided by m (with remainders). But since $r_{k+1} = r_{n+1}$ it will then follow that

$$d_{k+2} = d_{n+2} \text{ and } r_{k+2} = r_{n+2}$$

and so on (this could be proved by induction, but I think it is clear). Thus when the remainder repeats for the first time, the decimal repeats! How do we know that the remainders eventually repeat? *Because all remainders are between* 0 *and* $m - 1$. So by the time we have done m divisions with remainders, we must have come across a repeat of the remainders!!

Example: The expansion of 1/35 really does repeat as indicated above.

$1 = 35(0) + 1$ ($q = 0$ and $r = 1$) Decimal so far: 0

$10(1) = 35(0) + 10$ ($d_1 = 0$ and $r_1 = 10$) Decimal so far: 0.0

$10(10) = 35(2) + 30$ ($d_2 = 2$ and $r_2 = 30$) Decimal so far: 0.02

$10(30) = 35(8) + 20$ ($d_3 = 8$ and $r_3 = 20$) Decimal so far: 0.028

$10(20) = 35(5) + 25$ ($d_4 = 5$ and $r_4 = 25$) Decimal so far: 0.0285

$10(25) = 35(7) + 5$ ($d_5 = 7$ and $r_5 = 5$) Decimal so far: 0.02857

$10(5) = 35(1) + 15$ ($d_6 = 1$ and $r_6 = 15$) Decimal so far: 0.028571

$10(15) = 35(4)+10$ ($d_7 = 4$ and $r_7 = 10 = r_1$. Repeat!) Decimal: $0.0\overline{285714}$.

Remarks: The proposition is again telling us something we've already learned. On the other hand, now we've proved it! Also notice that the proposition tells us that any non-repeating decimal gives a real number which is not rational. My personal favorite has a simple pattern, but not a repeating one:

$$1.01001000100001000001\cdots$$

There are "many more" non-repeating decimals than repeating ones! This may seem a strange statement to make since there are infinitely many of both. One good way to think about this is that if a decimal could be chosen at random, then the chances of it repeating are less than the chances of winning the biggest lottery you could imagine!

Addition: We could try to define the addition of real numbers as an addition of infinite decimals, but this would be messy, as such an addition will typically involve infinitely many "carries." Instead, we'll define addition geometrically, via translations.

If r is a real number (possibly negative or 0), then **translation by r** is the slide of the number-line that is required to move 0 to r. Thus, for instance, translation by 1 slides the number-line one unit to the right, and translation by -1 slides it one unit to the left.

Translation definition of addition: If s is a real number, then $s + r$ is the resting place of s after translating it by r.

This sounds fancy, but I claim that it does the same thing as our earlier definitions of addition for integers. Why? Because the "next" integer after a is the translation of a by 1 unit to the right, and the "previous" integer before a is the translation of a by 1 unit to the left, while translating by 0 does nothing (induction takes care of the rest). It takes a bit more work to see:

Proposition 1.3.3. *The translation definition of addition does the same thing to* **rational numbers** *as the earlier definition.*

Proof: First of all, notice that the line of slope $\frac{1}{n}$ meets the line $x = 1$ at a point whose y-coordinate is "one nth of the way to 1." That is, it takes n of the translations by $\frac{1}{n}$ to move 0 to 1. This is seen by considering the triangle with vertices $(0, 0), (n, 0), (n, 1)$ and the similar triangle cut out by the line $x = 1$. Since it takes n (horizontal) translations by 1 to move from 0 to n, similar triangles tell us that the same is true of the vertical translations. We will refer to $\frac{1}{n}$ as a "fractional unit."

The sum of two rational numbers was:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

and we can assume that the fractions are in lowest terms, so $b > 0$ and $d > 0$. We can then put the fractions over a **common denominator:**

$$\frac{a}{b} = \frac{ad}{bd} \quad \text{and} \quad \frac{c}{d} = \frac{bc}{bd}$$

using Proposition 1.2.4. Now, translation by $\frac{c}{d}$ is translation by c of the fractional units $\frac{1}{d}$, which is also translation by bc of the fractional units $\frac{1}{bd}$, and likewise for $\frac{a}{b}$. Thus, addition of these rational numbers is translation of 0 by $ad+bc$ *of the fractional units* $\frac{1}{bd}$, which agrees with the addition definition above for rationals.

Laws of Addition: One could prove these with Euclidean geometry, but I would rather remind you that calculus does the job. The necessary ingredients are:

(a) The translation definition of addition is **continuous** and

(b) Every real number is a limit of rational numbers

because with these two ingredients, we can use the laws for addition of rational numbers to deduce the laws for the addition of real numbers. For example, real numbers r, s, t are limits of rational numbers:

$$r = \lim_{n\to\infty} \left\{ \frac{a_n}{b_n} \right\}, s = \lim_{n\to\infty} \left\{ \frac{c_n}{d_n} \right\}, t = \lim_{n\to\infty} \left\{ \frac{e_n}{f_n} \right\}$$

and because addition is continuous, we can pull out limits(!)

$$\begin{aligned}(r+s)+t &= \left(\lim_{n\to\infty} \left\{ \frac{a_n}{b_n} \right\} + \lim_{n\to\infty} \left\{ \frac{c_n}{d_n} \right\} \right) + \lim_{n\to\infty} \left\{ \frac{e_n}{f_n} \right\} \\ &= \lim_{n\to\infty} \left\{ \left(\frac{a_n}{b_n} + \frac{c_n}{d_n} \right) + \frac{e_n}{f_n} \right\}\end{aligned}$$

and because addition is associative for rationals we can substitute:

$$\left(\frac{a_n}{b_n} + \frac{c_n}{d_n} \right) + \frac{e_n}{f_n} = \frac{a_n}{b_n} + \left(\frac{c_n}{d_n} + \frac{e_n}{f_n} \right)$$

for each n, and plug back into the limits to get:

$$(r+s)+t = r+(s+t)$$

From the point of view of translations, it is obvious that:

0 is the additive identity and we can get mileage out of the:

Negation Transformation: This is defined the same way as before!

$$- : \mathbb{R} \to \mathbb{R}$$

takes a translation to the equal translation in the opposite direction. From this it is clear that $-r$ is the additive inverse of r, and arguing as in Proposition 1.2.3, we conclude that the negation transformation is a linear transformation: $-(r+s) = -r + (-s)$.

Subtraction is defined as usual, to be addition of the additive inverse.

Area definition of multiplication: For positive reals r and s, define:

$$rs \text{ is the } \textbf{area} \text{ of the } r \times s \text{ rectangle}$$

and then $r(-s) = -(rs)$, $(-r)s = -(rs), (-r)(-s) = rs$ and $r \cdot 0 = 0 = 0 \cdot r$.

Again, I am appealing to your geometric intuition of the meaning of area. It can be carefully defined using limits and calculus, if you prefer.

Proposition 1.3.4. *The area definition for real numbers does the same thing to* **rational numbers** *as the earlier definition.*

Proof: Because the definitions incorporate negatives in the same way, it is enough to see that the definitions are the same for positive rational numbers. Of course, the area of an $m \times 1$ rectangle is m. The area of an $m \times (n+1)$ rectangle is m more than the area of an $m \times n$ rectangle because an $m \times (n+1)$ rectangle is the union of $m \times n$ and $m \times 1$ rectangles! Thus the area definition agrees with the earlier definition for natural numbers.

As for positive rational numbers, it again comes down to the fact that n of the fractional $\frac{1}{n}$ units are equal to 1 unit. From this it follows that mn of the $\frac{1}{m} \times \frac{1}{n}$ squares exactly fill a 1×1 square, so $\frac{1}{mn} = \frac{1}{m} \times \frac{1}{n}$ in both definitions, and again we see that $\frac{k}{m} \times \frac{l}{n}$ is kl of the fractional squares $\frac{1}{mn}$, so it has the appropriate area $\frac{kl}{mn}$.

The rest of the laws of arithmetic have a very pretty geometric interpretation:

The Distributive Law:

$$r(s+t) = rs + rt$$

because an $r \times (s+t)$ rectangle is a union of $r \times s$ and $r \times t$ rectangles.

The Commutative Law:

$$rs = sr$$

because an $r \times s$ rectangle has the same area as an $s \times r$ rectangle.

The Associative Law:

$$r(st) = (rs)t$$

because both are the **volumes** of an $r \times s \times t$ **box**.

1 is the multiplicative identity. The area of an $r \times 1$ rectangle is r.

We finally want to prove that every real number except 0 has a multiplicative inverse. This can be done either using calculus or using geometry. For the calculus approach, let r be a positive real number. Then the function:

$$f(x) = rx$$

is continuous (in fact, differentiable with derivative $f'(x) = r$). Since $f(0) = 0$ and $\lim_{x \to +\infty} f(x) = +\infty$, the **intermediate value theorem** tells us that there must be some positive real number s so that $f(s) = 1$. In other words, $rs = 1$ so $s = 1/r$. And then, of course, $-s$ is the multiplicative inverse of $-r$.

The calculus proof only says that the inverse exists, not how to find it. For a geometric construction of the multiplicative inverse, let L be the line through $(0, 0)$ and $(r, 1)$. This has slope $1/r$ (unless $r = 0$, in which case it is vertical!). In particular, the intersection of L with the vertical line $x = 1$ is the point $(1, 1/r)$. That is, by drawing L and intersecting with $x = 1$, we have constructed the multiplicative inverse. This is more satisfying that just proving that it exists!

So $\mathbb{R}$ is a field.

As we said earlier, addition and multiplication of decimals is messy. There are, however, a couple of useful exceptions to this.

Multiplying by powers of 10: If $r = q.d_1d_2d_3\cdots$, then:

$$10r = (10q + d_1).d_2d_3d_4\cdots$$

$$100r = (100q + 10d_1 + d_2).d_3d_4d_5\cdots$$

etc.

That is, multiplying by powers of 10 shifts the decimal point.

Subtracting "matching" digits: Suppose r and s are real numbers with matching digits. That is, suppose:

r has decimal expansion $q.d_1d_2d_3\cdots$ and

s has decimal expansion $p.d_1d_2d_3\cdots$

Then

$$r - s = q - p$$

That is, if the decimals all match, then the difference is an integer.

In Proposition 1.3.2, we proved that every rational number expands as a repeating decimal. Here we prove the *converse* statement.

Proposition 1.3.5. *Every repeating decimal is the decimal expansion of some rational number.*

Proof: Start with a repeating decimal $r = q.d_1d_2...d_k\overline{d_{k+1}\cdots d_n}$. Then:

$$10^kr = (10^kq + 10^{k-1}d_1 + \cdots + d_k).\overline{d_{k+1}\cdots d_n}$$

and

$$10^nr = (10^nq + 10^{n-1}d_1 + \cdots + d_n).\overline{d_{k+1}...d_n}$$

and these are matching decimals, so we can subtract them to get:

$$10^nr - 10^kr = (10^nq + ... + d_n) - (10^kq + ... + d_k)$$

and dividing both sides by $10^n - 10^k$, we see that r is rational:

$$r = \frac{(10^nq + ... + d_n) - (10^kq + ... + d_k)}{10^n - 10^k}$$

Example: To find the rational number that expands to:

$$1.11\overline{12}$$

we take:

$$\frac{11112 - 111}{10^4 - 10^2} = \frac{11001}{9900} = \frac{3667}{3300}$$

Finally, from the irrationality of $\sqrt{2}$ (see §1.2) we get an interesting:

Corollary: The decimal expansion of $\sqrt{2}$ doesn't repeat!

1.3.1 Real Number Exercises

3-1 (a) Find the first 5 decimals in the expansion of $\sqrt{3}$ by squaring.

(b) Find the first 5 decimals in the expansion of $\sqrt[3]{2}$ by cubing.

3-2 Find the decimal expansions for each of the following:

(a) $\frac{1}{13}$ (b) $\frac{2}{13}$ (c) $\frac{3}{13}$ (d) $\frac{4}{13}$ (e) $\frac{5}{13}$ (f) $\frac{6}{13}$ (g) $\frac{7}{13}$

Do you see a pattern?

3-3 Convert each repeating decimal to a fraction in lowest terms.

(a) $0.\overline{27}$ (b) $0.0\overline{27}$ (c) $0.2\overline{27}$ (d) $0.\overline{027}$ (e) $0.\overline{037}$

3-4 Which rational numbers correspond to terminating decimals?

Infinite decimals are not the "most efficient" way to express a positive real number as a limit of rational numbers. The **continued fraction** expansion actually works much better.

Continued Fraction Expansion: Given a positive real number r, define its *continued fraction* by induction:

(i) q_1 is chosen so that $q_1 \leq r < q_1 + 1$ (q_1 is the **integer part** of r) and

$s_1 = r - q_1$ (s_1 is the **fractional part** of r, with $0 \leq s_1 < 1$).

(ii) Once s_n is defined, then if $s_n = 0$, STOP! Otherwise,

q_{n+1} is defined to be the integer part of $1/s_n$ and:

s_{n+1} is defined to be the fractional part of $1/s_n$.

This gives a sequence of rational numbers converging to r:

$$q_1,\ q_1 + \frac{1}{q_2},\ q_1 + \frac{1}{q_2 + \frac{1}{q_3}},\ q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \frac{1}{q_4}}}, \cdots$$

Example: Expand $25/17 = 1.\overline{4705882352941176}$ as a continued fraction.

$q_1 = 1$ and $s_1 = 8/17$ (so $1/s_1 = 17/8$),

$q_2 = 2$ and $s_2 = 1/8$ (so $1/s_2 = 8$),

$q_3 = 8$ and $s_3 = 0$. STOP! This gives the sequence:

$$1,\ 1 + \frac{1}{2} = \frac{3}{2},\ 1 + \frac{1}{2 + \frac{1}{8}} = \frac{25}{17}$$

Continued fraction expansions of rational numbers always terminate. (Why?)

3-5 Expand each of the following as continued fractions and write the sequences (as in the example above):

$$\text{(a)}\ \frac{56}{55} \qquad \text{(b)}\ \frac{57}{55} \qquad \text{(c)}\ \frac{59}{55} \qquad \text{(d)}\ \frac{89}{55}$$

Another Example: Expand $\sqrt{2} = 1.414\cdots$ as a continued fraction.

$q_1 = 1$ and $s_1 = \sqrt{2} - 1$

Next step. Simplify:

$$\frac{1}{s_1} = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1 = 2.414\cdots$$

$q_2 = 2$ and $s_2 = \sqrt{2} + 1 - 2 = \sqrt{2} - 1$ (same as s_1)

$q_3 = 2$ and $s_3 = \sqrt{2} - 1$ (same as q_2 and s_2)

so $2 = q_2 = q_3 = q_4 = \cdots$, giving the sequence:

$$1,\ 1 + \frac{1}{2} = \frac{3}{2},\ 1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5},\ 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = \frac{17}{12}, \cdots$$

In particular, the continued fraction for $\sqrt{2}$ doesn't terminate (this is another proof that $\sqrt{2}$ isn't rational!). But the q's do repeat. In fact, continued fraction expansions of solutions to the quadratic formula:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{with} \quad a, b, c \in \mathbb{Z}$$

(see §2.3) always repeat. (Why?)

3-6 Expand the following two numbers as continued fractions, indicating where the repeat in the q's occurs, and write out the first four terms of the sequence, as in the example.

(a) $\sqrt{3}$ (b) the golden mean: $\frac{1+\sqrt{5}}{2}$

Hint: The golden mean satisfies the cool property:

$$\frac{1}{\frac{1+\sqrt{5}}{2}-1}=\frac{1+\sqrt{5}}{2}$$

3-7 Calculator exercise. Use a calculator to find the first 5 values $q_1, q_2, ..., q_5$ in the continued fraction expansion of π and then find the rational number:

$$q_1+\frac{1}{q_2+\frac{1}{q_3+\frac{1}{q_4+\frac{1}{q_5}}}}$$

which is an excellent (much better than 3.14159) approximation of π.