Chapter 3

Symmetries

Symmetries are linear transformations of a vector space, so we will begin with a review of some linear algebra. In a basic linear algebra course, the scalars are real numbers, but here they might belong to any field. Each **algebraic number** (root of a polynomial with rational number coefficients) comes equipped with a multiplication transformation, which is a symmetry whose characteristic polynomial is the polynomial that gave birth to the algebraic number in the first place. The **constructible numbers** (numbers that can be constructed using only a compass and straightedge) are particular algebraic numbers that were of great interest to the ancients, and we will use linear algebra to understand them in some detail. Symmetries come in **groups**, and finite symmetry groups are always subgroups of **permutation** groups. There is a qualitative difference between permutation groups of 4 or less objects versus permutation groups of 5 or more objects, which we will explore. Finally, we will look at the **Galois group** of symmetries of the "splitting field" of an algebraic number, which is one of the most interesting objects in all of mathematics.