1.4 The Complex Numbers.

We start with an important property of the real numbers.

Proposition 1.4.1. *Every positive real number r has a single positive nth root for each natural number n. In other words, the equations*

$$x^n = r$$

each have exactly one positive real solution, which is denoted $\sqrt[n]{r}$.

Proof: The function:

$$f(x) = x^n$$

is continuous and differentiable, with derivative $f'(x) = nx^{n-1}$. Since $f(0) = 0$ and $\lim_{x\to+\infty} f(x) = +\infty$, the intermediate value theorem tells us that the graph of f crosses the line $y = r$ somewhere, say at the point (s, r). This means $f(s) = r$, or in other words $s^n = r$. But now we ask: "Why doesn't the graph of f cross $y = r$ **more than once?** Of course, it may cross the line $y = r$ again when x is negative (if n is even). But when x is positive, then $f(x)$ is a strictly **increasing** function because $f'(x) = nx^{n-1} > 0$. And strictly increasing functions cannot take the same value more than once!

It may seem as though with the real numbers we have reached the ultimate number system. However, the fact that negative real numbers do not have real square roots leads us to one final improvement.

The "purely imaginary number" i is by definition a square root of -1:

$$i^2 = -1$$

To be honest, this doesn't seem much more "imaginary" to me than the negative numbers, which were introduced in order to have additive inverses. Just as it made good geometric sense to place -1 one unit to the left of 0 on the number-line, it turns out to make good geometric sense to place i one unit above 0 on a "number-plane".

The Complex Numbers:

$$\mathbb{C} = \{\text{points on the number-plane}\}$$

A point in the plane is given by two real coordinates (s, t) or else as:

$$s + ti$$

The complex numbers are no longer ordered, since it makes no sense any more to write: $s + ti < u + vi$, but addition and multiplication can still be defined, so that $\mathbb{C}$ a field, and what is more, addition and multiplication have very useful geometric interpretations.

Definition of Addition. Addition of complex numbers is defined to be vector addition, $(s,t)+(u,v)=(s+u,t+v)$, which can also be written:

$$(s+ti)+(u+vi)=(s+u)+(t+v)i$$

Vector addition takes the translation definition for addition of real numbers and promotes it to a translation definition for the addition of vectors in spaces of all dimensions. In the case of the complex numbers, the space is two-dimensional. But the rules for addition will hold in all dimensions:

Addition is associative:

$$((s,t)+(u,v))+(x,y)=((s+u)+x,(t+v)+y))$$

$$(s,t)+((u,v)+(x,y))=(s+(u+x),t+(v+y))$$

These are the same because addition of real numbers is associative. Similarly,

Addition is commutative, and (0,0) is the additive identity.

We also have a fancier:

Negation Transformation: This time the negation transformation

$$-:\mathbb{C}\to\mathbb{C}$$

takes (s,t) to $(-s,-t)$. It reflects the number-plane across the origin.

Definition of Multiplication: Multiplication of complex numbers is defined by $(s,t)\cdot(u,v)=(su-tv,sv+tu)$, which may also be written:

$$(s+ti)(u+vi)=(su-tv)+(sv+tu)i$$

Unlike addition, this is **not** something we get for free by thinking of $\mathbb{C}$ as a vector space. Instead, this definition is forced upon us by the distributive law, and the fact that $i^2=-1$.

Multiplication is commutative:

$$(s+ti)(u+vi)=su-tv+(sv+tu)i$$

$$(u+vi)(s+ti)=us-vt+(ut+vs)i$$

These are the same because multiplication of real numbers is commutative!

Multiplication is associative: Check this for yourself.

Multiplication distributes with addition: (Exercise.)

(1,0) is the multiplicative identity:

$$(1+0i)(u+vi)=(u-0)+(v+0)i=u+vi$$

Finally, we want multipicative inverses. To do this, we introduce a second:

Conjugation Transformation: This is the function:

$$c : \mathbb{C} \to \mathbb{C}$$

such that $c(s + ti) = \overline{s + ti} = s - ti$. It reflects the plane across the x-axis. Notice that only the real numbers are unchanged under conjugation, and that the "purely imaginary" numbers ti conjugate to their additive inverses $-ti$.

Proposition 1.4.2. *The conjugation transformation is both linear* **and** *multiplicative. That is:*

$$\overline{(s+ti)} + \overline{(u+vi)} = \overline{(s+ti) + (u+vi)} \text{ and}$$
$$\overline{(s+ti)} \cdot \overline{(u+vi)} = \overline{(s+ti)(u+vi)}$$

Proof: Let's work them out:

$\overline{(s+ti)+(u+vi)} = \overline{(s+u)+(t+v)i} = (s+u)-(t+v)i.$

$\overline{(s+ti)} + \overline{(u+vi)} = (s-ti)+(u-vi) = (s+u)-(t+v)i.$ Check.

$\overline{(s+ti)(u+vi)} = \overline{(su-tv)+(sv+tu)i} = (su-tv)-(sv+tu)i.$

$\overline{(s+ti)} \cdot \overline{(u+vi)} = (s-ti)(u-vi) = (su-tv)-(sv+tv)i.$ Check.

It is somewhat surprising that conjugation is a multiplicative transformation! After all, the negation transformation certainly isn't multiplicative:

$$(-r)(-s) = rs, \text{ not } -(rs)$$

Absolute Value: The absolute value of a complex (or real) number is its Euclidean distance from $0 = (0, 0)$. That is,

$$|s + ti| = \sqrt{s^2 + t^2}$$

and it is very useful to notice that:

$$|s + ti|^2 = s^2 + t^2 = (s + ti)(s - ti) = (s + ti)\overline{(s + ti)}$$

and that whenever $s + ti \neq 0$, then:

$$1 = \frac{s^2+t^2}{s^2+t^2} = \frac{(s+ti)(s-ti)}{s^2+t^2} = (s+ti)\left(\frac{s-ti}{s^2+t^2}\right)$$

so that $s + ti$ **has a multiplicative inverse**, namely:

$$\frac{1}{(s+ti)} = \frac{s-ti}{s^2+t^2} = \frac{s}{s^2+t^2} - \frac{t}{s^2+t^2}i$$

Thus:

$\mathbb{C}$ **is a field!**

I promised a useful geometric interpretation of the multiplication of complex numbers. This is done using:

Polar Coordinates: If r is a positive real number (or zero) and θ is any real number, then the "polar coordinates":

$$(r;\theta)$$

are the coordinates of the unique point in the plane which is at the distance r from 0, and such that the line segment between 0 and $(r;\theta)$ is at the angle θ from the positive x-axis (measured counter-clockwise). I have put a semi-colon between r and θ to distinguish this notation from the vector notation for a point in the plane. Also **all angles will be measured in radians.**

There is some redundancy in polar coordinates. Precisely:

$(0;\theta)$ is the origin whatever θ may be, and

$(r;\theta)$ and $(r;\theta+2\pi a)$ are the same point when a is an integer.

Proposition 1.4.3. *In polar coordinates, the multiplication rule for complex numbers becomes:*

$$(r;\theta)\cdot(s;\psi)=(rs;\theta+\psi)$$

which is a wonderfully simple geometric description. In English:

Multiplication Rule: *To multiply two complex numbers in polar coordinates, add their angles and multiply their distances from* 0.

Proof: Recall that $\cos(\theta)$ and $\sin(\theta)$ are the x and y-coordinates of the point on the unit circle at the angle θ from the positive real axis. Thus,

$$(1;\theta)=(\cos(\theta),\sin(\theta))$$

and replacing 1 by r multiplies through by r, so $(r;\theta)=(r\cos(\theta),r\sin(\theta))$.

To see the rule, we translate from polar to vector coordinates, do the multiplication, and then translate back into polar coordinates. Let $(r;\theta)$ and $(t;\psi)$ be our two complex numbers. Then:

$$(r;\theta)\cdot(t;\psi)=(r\cos(\theta),r\sin(\theta))\cdot(t\cos(\psi),t\sin(\psi))$$

$$=(r\cos(\theta)t\cos(\psi)-r\sin(\theta)t\sin(\psi),r\cos(\theta)t\sin(\psi)+r\sin(\theta)t\cos(\psi))$$

$$=(rt(\cos(\theta)\cos(\psi)-\sin(\theta)\sin(\psi)),rt(\cos(\theta)\sin(\psi)+\sin(\theta)\cos(\psi)))$$

Now, remember the angle addition identities from trigonometry:

$$\cos(\theta+\psi)=\cos(\theta)\cos(\psi)-\sin(\theta)\sin(\psi)$$

and

$$\sin(\theta+\psi)=\cos(\theta)\sin(\psi)+\sin(\theta)\cos(\psi)$$

Substituting these identities into our formula for the product gives:

$$(r;\theta)\cdot(t;\psi)=(rt\cos(\theta+\psi),rt\sin(\theta+\psi))$$

and in polar coordinates, this is: $(rt;\theta+\psi)$. Done!

Corollary 1.4.4. *There are n different nth roots of any complex number except for 0, which always has only one nth root!*

Proof: Let $z = (r; \theta)$ in polar coordinates. Taking an nth power is easy to do using Proposition 1.4.3. Namely:

$$z^n = (r; \theta)^n = (r^n; n\theta)$$

But then using Proposition 1.4.1, we see that conversely:

$$\left(\sqrt[n]{r}; \frac{\theta}{n}\right)$$

is an nth root of $(r; \theta)$ (we use Proposition 1.4.1 for $\sqrt[n]{r}$). But:

$$\left(\sqrt[n]{r}; \frac{\theta}{n} + \frac{2\pi}{n}\right), \left(\sqrt[n]{r}; \frac{\theta}{n} + \frac{4\pi}{n}\right), \ldots, \left(\sqrt[n]{r}; \frac{\theta}{n} + \frac{(2n-2)\pi}{n}\right)$$

are also nth roots of r, and what's more, these are all **different** complex numbers because the angles between any two of them do not differ by a multiple of 2π. We have n of these in all, so we have found as many nth roots as we wanted. But why aren't there any more? As we've seen, if $(s; \psi)$ is an nth root of $(r; \theta)$, then $s = \sqrt[n]{r}$ must be the **unique** positive nth root of r from Proposition 1.4.1. Moreover, the n angles we've listed above are the **only** angles between 0 and 2π with the property that $n\psi = \theta + 2\pi a$. This tells us that if $(s; \psi)$ is any complex number other than the n roots listed above, then either $s^n \neq r$ or else $n\psi \neq \theta + 2\pi a$. Thus, there are no more nth roots than these!

Examples: (a) The two square roots of $i = (1; \frac{\pi}{2})$ are:

$$\left(1; \frac{\pi}{4}\right) \text{ and } \left(1; \frac{\pi}{4} + \frac{2\pi}{2}\right) = \left(1; \frac{5\pi}{4}\right)$$

In ordinary complex number notation, these are: $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

(b) The three cube roots of $-8 = (8; \pi)$ are:

$$\left(2; \frac{\pi}{3}\right), \left(2; \frac{\pi}{3} + \frac{2\pi}{3} = \pi\right), \text{ and } \left(2; \frac{\pi}{3} + \frac{4\pi}{3} = \frac{5\pi}{3}\right)$$

which in ordinary notation for complex numbers are: $1 + \sqrt{3}i$, -2 and $1 - \sqrt{3}i$.

Remark: We introduced i as an imaginary square root of -1, used it to define the complex numbers, and now we see that by doing this, we have in fact given ourselves **all** possible nth roots of **all** numbers, even the new complex numbers themselves! This is indeed a remarkable development. But it gets even better. In §2.5, we will see that all roots of all polynomials with complex number coefficients (not just the polynomials $x^n = z$) are complex numbers.

I can't resist finishing by pointing out the link between:

Exponentials and Trigonometry: The Taylor series for e^z is given by:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + ...$$

and this series, which converges for all real numbers, also converges for all **complex** numbers. Moreover, if $z = i\theta$ is a purely imaginary complex number, then:

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \tfrac{(i\theta)^2}{2!} + \tfrac{(i\theta)^3}{3!} + \tfrac{(i\theta)^4}{4!} + ... \\ &= 1 + i\theta - \tfrac{\theta^2}{2!} - i\tfrac{\theta^3}{3!} + \tfrac{\theta^4}{4!} + i\tfrac{\theta^5}{5!} - ... \\ &= (1 - \tfrac{\theta^2}{2!} + \tfrac{\theta^4}{4!} - ...) + i(\theta - \tfrac{\theta^3}{3!} + \tfrac{(\theta)^5}{5!} - ...) \end{aligned}$$

and then the Taylor series for $\cos(\theta)$ and $\sin(\theta)$ tell us that:

$$e^{i\theta} = \cos(\theta) + \sin(\theta)i$$

which is even simpler in polar coordinates:

$$e^{i\theta} = (1; \theta).$$

We can multiply through by a positive real number r, to get:

$$re^{i\theta} = (r; \theta)$$

This gives a new way of looking at Proposition 1.4.3:

$$(r; \theta) \cdot (t; \psi) = (re^{i\theta}) \cdot (te^{i\psi}) = rte^{i\theta}e^{i\psi} = rte^{i(\theta+\psi)} = (rt; \theta + \psi)$$

so the angle addition is just the rule for exponents: $e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)}$.

Since $(1; \pi) = -1$, this interpretation of polar coordinates gives:

$$e^{i\pi} = -1$$

which is an extraordinary relation among the special numbers: i, π, e and -1.

1.4.1 Complex Number Exercises

4-1 For each of the following complex numbers:

(a) $3 + 4i$, (b) $3 - 4i$ (c) $-3 + 4i$ (d) $-3 - 4i$

(i) Square it. (ii) Find its multiplicative inverse.

(iii) Find (approximate) polar coordinates for it.

(iv) Find both square roots of it, in both polar and rectangular coordinates.

(v) Plot it, its inverses and its square roots.

Note: To put $s + ti$ in polar coordinates, set:

$$r = \sqrt{s^2 + t^2} \text{ and } \theta = \tan^{-1}(t/s)$$

There is a subtlety in determining the angle θ, though. If, for example, you feed $-1 - i$ into your calculator (always set for **radians!**), it will give you:

$$r \approx 1.414 \text{ and } \theta \approx 0.7854$$

which are the approximate polar coordinates for $1 + i$, not for $-1 - i$. The problem is that the calculator always chooses $\tan^{-1}$ so that the angle is between $-\pi/2$ and $\pi/2$. In other words, it will always assume that $s \geq 0$. If your complex number has a negative value of s, you will need to add π to the value of θ given by your calculator to get the "true" θ. Thus:

$$r \approx 1.414 \text{ and } \theta \approx \pi + 0.7854 \approx 3.927$$

are the true approximate polar coordinates for $-1 - i$.

4-2 Find $(1 + 2i)^5$ and $(1 + 2i)^{10}$ in two different ways:

(a) Multiply them out cleverly (show your work!).

(b) Convert to (approximate) polar coordinates, take the power, then convert back to rectangular coordinates.

4-3 Prove the distributive law for complex numbers.

4-4 Prove the following:

(a) $|(s + ti)(u + vi)| = |s + ti||u + vi|$

(b) $|1/(s + ti)| = 1/|s + ti|$.

(c) $\overline{1/(s + ti)} = 1/(\overline{s + ti})$

4-5 Find the polar coordinates for each of the following:

(a) $\overline{(r; \theta)}$, (b) $-(r; \theta)$ (c) $1/(r; \theta)$ (d) $-\overline{1/(r; \theta)}$

4-6 (a) Find all the eighth roots of 16 in exact rectangular coordinates.

(b) Find all the twelfth roots of 16 in exact polar coordinates.

4-7 The **Gaussian integers** are:

$$\mathbb{Z}[i] = \{a + bi \text{ such that } a, b \in \mathbb{Z}\} \subset \mathbb{C}$$

The four Gaussian integers with multiplicative inverses are $1, -1, i, -i$. All the other Gaussian integers are "interesting." A Gaussian integer $a + bi$ is **prime** if its only factors are $1, -1, i, -i$ or one of these multiplied by $a + bi$, namely $a + bi, -a - bi, b - ai$ or $-b + ai$.

Notice:

$$2 + 0i = (1 + i)(1 - i) = 1^2 + 1^2 \quad \text{and} \quad 5 + 0i = (1 + 2i)(1 - 2i) = 1^2 + 2^2$$

so 2 and 5 are no longer primes when thought of as Gaussian integers!

(a) Graph all the Gaussian integers on a chunk of the number plane.

(b) For each of the integer primes from 2 to 30, decide whether they can be factored or remain prime when thought of as Gaussian integers. Can you detect a pattern?

(c) Would you expect 10007 to be a prime Gaussian integer or not?

(Hint: It has a remainder of 3 when divided by 4.)

4-8 The **Eisenstein integers are:**

$$\mathbb{Z}[\omega] \ = \{a + b\omega \ \text{ such that } \ a, b \in \mathbb{Z}\} \subset \mathbb{C}$$

where

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

(a) Show that the product of two Eisenstein integers is again an Eisenstein integer. (This was obvious for the Gaussian integers!)

(b) Show that

$$\overline{\omega} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

is an Eisenstein integer.

(c) Graph the Eisenstein integers on a chunk of the number-plane.

(d) Which six Eisenstein integers have multiplicative inverses that are also Eisenstein integers? These are the “uninteresting” ones.

(e) Calculate $(a + b\omega)(a + b\overline{\omega})$.

(f) Show that 3 is not prime as an Eisenstein integer.