University of Utah, Department of Mathematics
January 2018, Algebra Qualifying Exam

There are ten problems on the exam. You may attempt as many problems out of the 10 problems below as you wish; five correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.

1. Show that there is no simple group of order 112. You may use the fact that A_7 is simple.

2. Describe the automorphisms of the group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$.

3. Prove that there is no element in S_9 of order 18.

4. Let M be the $\mathbb{C}[x]$-module generated by elements a, b, c modulo the three relations $a + b, x^3 a + xb + xc$ and $(x^3 + 1)a + b$. Write M as a direct sum of cyclic $\mathbb{C}[x]$-modules.

5. Let $R = \mathbb{Q}[x, y, z]$ and let $I = (x, y)$ and $J = (y, z)$. Compute $\mathrm{Tor}_i^R(R/I, R/J)$ for all values of $i \geq 0$.

6. Find all prime ideals of the ring $\mathbb{Z}[x]/(15, x^3 - 2)$.

7. Prove that if a 3×3 matrix A over $\mathbb{Q}$ satisfies $A^8 = I$, then $A^4 = I$. Justify claims of irreducibility of polynomials.

8. Let $F \subset L$ a extension of fields of degree 4. Prove that there are no more than 3 fields proper intermediate subfields K; namely, such that $F \subset K \subset L$.

9. Compute the Galois group of the polynomial $x^{10} + x^5 + 1$ over $\mathbb{Q}$.

10. Let F be a field and n a positive integer such that F has no nontrivial field extensions of degree less than n. Let $L = F[\alpha]$ be an extension such that α^n is in F. Prove that each element of L is a product of elements of the form $a\alpha + b$, with a, b in F.