**University of Utah, Department of Mathematics**
**August 2019, Algebra Qualifying Exam**

*There are ten problems on the exam. You may attempt as many problems as you wish; five correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.*

1. Prove that the additive group of $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

2. Determine the splitting field of the polynomial $x^p - x - a$ over $\mathbb{F}_p$ where $0 \neq a \in \mathbb{F}_p$. Show that the Galois group is cyclic.

3. What is the Galois closure of $\mathbb{Q}(\sqrt{1+\sqrt{2}})/\mathbb{Q}$?

4. Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic field over $\mathbb{Q}$.

5. Find the Galois group of $f(x) = x^4 + 2x^2 + x + 3$.

6. Let $R = \mathbb{F}_3[x]$. Consider the $3 \times 3$ matrix

$$\begin{bmatrix} 0 & x & x \\ 1 & x & 1+x \\ 1+x^2 & x^2 & 1+x^2 \end{bmatrix}$$

   Let $\varphi : R^3 \longrightarrow R^3$ be the $R$-module homomorphism given by this matrix. Write the cokernel of $\varphi$ as a direct sum of cyclic modules.

7. Show that there is no simple group of order $552 = 23 \cdot 3 \cdot 2^3$.

8. Suppose that $k$ is the field with 3 elements and let $R = k[x]$. Identify nine different (up to isomorphism) $R$-modules $M$ such that $|M| = 9$.

9. Suppose $\varphi : G \longrightarrow H$ is a group homomorphism and $H$ is a group of order 33. Further suppose that $N \supseteq \ker \varphi$ is a subgroup of $G$. Prove that $N$ is normal.

10. Suppose $k$ is a field and suppose that $f(x) \in k[x]$ has degree $n$. Prove that $f(x)$ has no repeated factors in $k[x]$ if and only if all $n \times n$ matrices over $k$ with characteristic polynomial $f(x)$ are similar.