Abstract Algebra. Math 6310. Bertram/Utah 2022-23.

Derived Functors

Let $\mathcal{A}$ be an abelian category.

Definition. (a) $\mathcal{A}$ *has enough projectives* if each object A admits:

$$P \to A \to 0$$

an epimorphism from a projective object P of $\mathcal{A}$.

(b) $\mathcal{A}$ *has enough injectives* if each object A admits:

$$0 \to A \to I$$

a monomorphism to an injective object I of $\mathcal{A}$.

Fortunately for us, the categories $\mathcal{M}od_R$ of R-modules have enough of both. Note that by iterating, we obtain *exact complexes* of projectives and of injectives:

$$\cdots \to P_2 \stackrel{d_2}{\to} P_1 \stackrel{d_1}{\to} P_0 \stackrel{d_0}{\to} A \to 0$$

and

$$0 \to A \stackrel{d_0}{\to} I_0 \stackrel{d_1}{\to} I_1 \stackrel{d_2}{\to} I_2 \to \cdots$$

We will use the first to construct *left derived functors* (of a right-exact functor) and the second to construct *right derived functors* (of a left-exact functor).

Remark. One might instead use superscripts for the terms of the injective resolution (which is a "cochain" since the indices increase as one moves to the right).

Recall that given an R-module M, the Hom (covariant) functor:

$$F_M := \text{Hom}_R(M, \cdot) : \mathcal{M}od_R \to \mathcal{M}od_R$$

is left-exact. The opposite Hom functor $F^M = \text{Hom}_R(\cdot, M)$ is also left-exact, but behaves more like a right-exact functor since it is contravariant. The tensor product defines a right-exact covariant functor as follows:

Proposition 1. Tensoring with a fixed R-module M defines the functor:

$$T_M(N) = N \otimes_R M, \text{ with}$$

$$T_M(f : N \to N') = (f \otimes 1_M : N \otimes_R M \to N' \otimes_R M)$$

that is (covariant and) right-exact.

Proof. It is clear that this is a functor. Right-exactness is the issue. Let

$$(*)\ N \stackrel{f}{\to} N' \stackrel{g}{\to} N'' \to 0$$

be a right-exact sequence of R-modules. Then:

(i) $g \otimes 1_M$ is surjective (this is obvious).

(ii) $(g \otimes 1_M) \circ (f \otimes 1_M) = (g \circ f) \otimes 1_M = 0$ (this is also obvious)

(iii) The morphism $g \otimes 1_M$ is the cokernel of $f \otimes 1_M$. Recall the universal properties UC and UT of the cokernel (in an arbitrary abelian category) and tensor product (in the category of R-modules) respectively, and consider:

$$N \times M \stackrel{(f,1_M)}{\to} N' \times M \stackrel{(g,1_M)}{\to} N'' \times M \to 0$$

the sequence of R-bilinear maps, with the analogue of the cokernel property:

UC: Any bilinear map $b' : N' \times M \to L$ such that $b' \circ (f, 1_M) = 0$ is the composition $b'' \circ (g, 1_M)$ for the unique bilinear map $b'' : N'' \times M \to L$ defined by $b''(g(n'), m) = b'(n', m)$. Coupling this with the universal property UT of the tensor product, we obtain the following:

An R-module homomorphism $h' : N' \otimes_R M \to L$ with $h' \circ (f \otimes 1_M) = 0$ gives:

$$b' : N' \times M \to N' \otimes_R M \to L \text{ with } b' \circ (f, 1_M) = 0$$

which therefore factors through a unique bilinear map $b'' : N'' \times M \to L$ and, by UT, factors uniquely through an R-module homomorphism $h'' : N'' \otimes_R M \to L$.

Thus $g \otimes 1_M$ is the cokernel of $f \otimes 1_M$ in the abelian category $\mathcal{M}od_R$, which is to say that the sequence $(*) \otimes_R M$ is exact at the middle term. $\square$

Getting back to the projectives:

Proposition 2. In an arbitrary abelian category $\mathcal{A}$, suppose:

$$\begin{array}{ccccccccccc} \cdots & \to & P_2 & \stackrel{d_2}{\to} & P_1 & \stackrel{d_1}{\to} & P_0 & \stackrel{d_0}{\to} & A & \to & 0 \\ & & & & & & & & \downarrow f & & \\ \cdots & \to & E_2 & \stackrel{\partial_2}{\to} & E_1 & \stackrel{\partial_1}{\to} & E_0 & \stackrel{\partial_0}{\to} & A' & \to & 0 \end{array}$$

are two exact sequences, the first made up of projectives. Then:

(a) There is an extension of f to a morphism of chain complexes:

$$f_\bullet : P_\bullet \to E_\bullet$$

(b) Any two such extensions $f_\bullet$ and $g_\bullet$ are homotopic maps of chain complexes.

Proof. (a) The map $f \circ d_0 : P_0 \to A'$ lifts to $f_0 : P_0 \to E_0$ using the facts that ∂_0 is surjective and P_0 is projective. Then f_0 maps $\ker(d_0) = \text{im}(d_1)$ to $\ker(\partial_0) = \text{im}(\partial_1)$ since $f \circ d_0 = \partial_0 \circ f_0$ and so we may once more lift $f_0 \circ d_1$ to $f_1 : P_1 \to E_1$ satisfying $f_0 \circ d_1 = \partial_1 \circ f_1$ and continue.

(b) Given two such extensions $f_\bullet$ and $g_\bullet$ each making the diagram commute:

$$\begin{array}{ccccccccccc} \cdots & \to & P_2 & \stackrel{d_2}{\to} & P_1 & \stackrel{d_1}{\to} & P_0 & \stackrel{d_0}{\to} & A & \to & 0 \\ & & f_2 \downarrow\downarrow g_2 & & f_1 \downarrow\downarrow g_1 & & f_0 \downarrow\downarrow g_0 & & f \downarrow\downarrow f & & \\ \cdots & \to & E_2 & \stackrel{\partial_2}{\to} & E_1 & \stackrel{\partial_1}{\to} & E_0 & \stackrel{\partial_0}{\to} & A' & \to & 0 \end{array}$$

we also define the homotopy between them inductively. First, we let:

$$0 = h_{-1} : A \to E_0 \text{ so that } f - f = \partial_0 \circ h_{-1}$$

Then we notice that $f_0 - g_0$ maps P_0 to the kernel of ∂_0, so we may choose:

$$h_0 : P_0 \to E_1 \text{ so that } f_0 - g_0 = \partial_1 \circ h_0 = \partial_1 \circ h_0 + h_{-1} \circ d_0$$

Then $\partial_1(f_1 - g_1 - h_0 \circ d_1) = (f_0 - g_0) \circ d_1 - (\partial_1 \circ h_0) \circ d_1 = 0$, so we choose:

$$h_1 : P_1 \to E_2 \text{ so that } f_1 - g_1 - h_0 \circ d_1 = \partial_2 \circ h_1$$

and one more step gets us to the general case. We have $\partial_2(f_2 - g_2 - h_1 \circ d_2) = (f_1 - g_1) \circ d_2 - (\partial_2 \circ h_1) \circ d_2 = (f_1 - g_1) \circ d_2 - (f_1 - g_1 - h_0 \circ d_1) \circ d_2 = 0$ and this allows us to choose:

$$h_2 : P_2 \to E_3 \text{ so that } f_2 - g_2 - h_1 \circ d_2 = \partial_3 \circ h_2$$

and off we go. In the end, we have the desired homotoy $h_i : P_i \to E_{i+1}$ satisfying:

$$f_i - g_i = \partial_{i+1} \circ h_i + h_{i-1} \circ d_i \quad \square$$

Corollary. Let $F : \mathcal{A} \to \mathcal{B}$ be a right-exact functor of abelian categories and assume $\mathcal{A}$ has enough projectives. Then the sequence of left derived functors:

$$L_iF(A) := H_i(F(P_\bullet));\ L_iF(f : A \to B)) = H_i(F(f) : F(P_\bullet) \to F(Q_\bullet))$$

are well-defined (only up to isomorphism, unfortunately) by choosing projective resolutions $P_\bullet$ (for A) and $Q_\bullet$ (for B) and using Proposition 2.

Proof. We show any two projective resolutions of A give isomorphic homologies:

$$H_i(F(P_\bullet)) \text{ and } H_i(F(P'_\bullet))$$

To this end, we apply the Proposition twice to get:

$$\begin{array}{ccccccccc}
\cdots & \to & P_2 & \stackrel{d_2}{\to} & P_1 & \stackrel{d_1}{\to} & P_0 & \stackrel{d_0}{\to} & A & \to & 0 \\
 & & \downarrow i_2 & & \downarrow i_1 & & \downarrow i_0 & & \downarrow 1_A & & \\
\cdots & \to & P'_2 & \stackrel{d'_2}{\to} & P'_1 & \stackrel{d'_1}{\to} & P'_0 & \stackrel{d'_0}{\to} & A & \to & 0 \\
 & & \downarrow j_2 & & \downarrow j_1 & & \downarrow j_0 & & \downarrow 1_A & & \\
\cdots & \to & P_2 & \stackrel{d_2}{\to} & P_1 & \stackrel{d_1}{\to} & P_0 & \stackrel{d_0}{\to} & A & \to & 0
\end{array}$$

and homotopies $h_i : P_i \to P_{i+1}$ exhibiting $j_\bullet \circ i_\bullet \sim 1_{P_\bullet}$ (from the Proposition since both sides are lifts of 1_A), and $h'_i : P'_i \to P'_{i+1}$ exhibiting $i_\bullet \circ j_\bullet \sim 1_{P'_\bullet}$.

Now we apply F to everything, and get morphisms $(F \circ i)_\bullet$ and $(F \circ j_\bullet)$ and homotopies $(F \circ h)_\bullet$ and $(F \circ h')_\bullet$ exhibiting $(F \circ j_\bullet) \circ (F \circ i_\bullet) \sim 1_{F \circ P}$ and $(F \circ i)_\bullet \circ (F \circ j)_\bullet \sim 1_{F \circ P'}$. Since homotopic maps of complexes induce the same maps on homology, it follows that each $H_i(F \circ i) : H_i(F(P)) \to H_i(F(P'))$ is an isomorphism, with inverse $H_i(F \circ j)$.

The Corollary then follows (except for the troubling isomorphism business) by applying Proposition 2 to $P_\bullet$ and $Q_\bullet$ and $f : A \to B$. ☐

Theorem 3. Given a right-exact functor $F : \mathcal{A} \to \mathcal{B}$ and a short-exact sequence:

$$(*)\ 0 \to A \to A' \to A'' \to 0$$

in an abelian category $\mathcal{A}$ with enough projectives, there is a long exact sequence:

$$\to L_1F(A') \to L_1F(A'') \to F(A) \to F(A') \to F(A'') \to 0$$

of objects of $\mathcal{B}$.

Proof. Choose projective resolutions: $P_\bullet \to A$ and $P''_\bullet \to A''$. We will fashion a third projective resolution of A' that fits in a short exact sequence:

$$0 \to P_\bullet \to P'_\bullet \to P''_\bullet \to 0$$

of chain complexes, which *remains* short-exact after applying the functor F to each of the complexes. The Zigzag Lemma then gives the desired long exact sequence among the homology objects of $F(P_\bullet)$, $F(P'_\bullet)$ and $F(P''_\bullet)$, which is the result.

In fact, the terms of $P'_\bullet$ are direct sums $P'_i = P_i \oplus P''_i$ and the horizontal maps are the standard inclusions and projections:

$$0 \to P_i \stackrel{\iota}{\to} P_i \oplus P''_i \stackrel{q}{\to} P''_i \to 0$$

which explains why these horizontal sequences remain exact after applying F.

In the diagram below, the map $d_0'' : P_0'' \to B''$ lifts (because P_0'' is a projective) to a map $P_0'' \to B'$, and then we obtain a commutative diagram:

$$\begin{array}{ccccccccc} & & P_0 & \to & P_0 \oplus P_0' & \to & P_0'' & & \\ & & \downarrow d_0 & \searrow & \downarrow d_0' & \swarrow & \downarrow d_0'' & & \\ 0 & \to & B & \stackrel{f}{\to} & B' & \stackrel{g}{\to} & B'' & \to & 0 \end{array}$$

with d_0' defined (and surjective by the five lemma...or rather the first four lemma!) using the universal property of the coproduct. Then we consider:

$$0 \to \ker(d_0) \to \ker(d_0') \to \ker(d_0'') \to 0$$

which is exact (by the snake lemma) giving a diagram just as the one above:

$$\begin{array}{ccccccccc} & & P_1 & \to & P_1 \oplus P_1' & \to & P_1'' & & \\ & & \downarrow d_1 & \searrow & \downarrow d_1' & \swarrow & \downarrow d_1'' & & \\ 0 & \to & \ker(d_0) & \to & \ker(d_0') & \to & \ker(d_0'') & \to & 0 \end{array}$$

with surjective vertical maps, etc. □

Example. The left derived functors of the tensor functor $T_M(N) = N \otimes_R M$ are:

$$\mathrm{Tor}_i^R(N, M) := L_i T_M(N)$$

Thus, for example, to compute $\mathrm{Tor}_i(M, k)$, we will use the (free) Koszul resolution:

$$0 \to k[x,y] \stackrel{(-y,x)}{\to} k[x,y] \oplus k[x,y] \stackrel{x+y}{\to} k[x,y] \to k \to 0$$

for k, and then we obtain $\mathrm{Tor}_i(M, k)$ as the homologies of the sequence:

$$M \stackrel{(-y,x)}{\to} M \oplus M \stackrel{x+y}{\to} M$$

(since $M \otimes_R R = M$). Thus, for instance when $M = k$, all maps are zero(!) and:

$$\mathrm{Tor}_2(k,k) = k, \ \mathrm{Tor}_1(k,k) = k^2 \text{ and } \mathrm{Tor}_0(k,k) = k \otimes_R k = k$$

When $M = k[y] = k[x,y]/\langle x \rangle$, only the x map is zero, and we get:

$$\mathrm{Tor}_2(k[y],k) = 0, \ \mathrm{Tor}_1(k[y],k) = k \text{ and } \mathrm{Tor}_0(k[y],k) = k \otimes_R k[y] = k$$

Or we could resolve $k[y]$ instead: $0 \to k[x,y] \stackrel{x}{\to} k[x,y] \to k[y] \to 0$ and then:

$$M \stackrel{x}{\to} M$$

computes $\mathrm{Tor}_i(M, k[y])$, so e.g. $\mathrm{Tor}_i(k, k[y]) = \mathrm{Tor}_i(k[y], k]$. This is no accident.

Finally, given the short-exact sequence:

$$0 \to k[y] \stackrel{y}{\to} k[y] \to k \to 0$$

we can get a long exact sequence of Tor's by applying the functor $\otimes k$. This gives:

$$0 \to \mathrm{Tor}_2(k,k) \to \mathrm{Tor}_1(k[y],k) \stackrel{0}{\to} \mathrm{Tor}_1(k[y],k) \to \mathrm{Tor}_1(k,k) \to k \stackrel{0}{\to} k \to k \to 0$$

Remark. If R is a PID, then every finitely generated module N resolves as:

$$0 \to R^m \to R^n \to N \to 0$$

for some free modules R^m and R^n. It follows that $\mathrm{Tor}_i(M \otimes N) = 0$ when $i > 1$. This is, in particular, the case for finitely generated abelian groups.

Meanwhile, over in Opposite Land...

By reversing all arrows and replacing projectives with injectives, we get:

3 Theorem: Given a left-exact functor $G : \mathcal{A} \to \mathcal{B}$ from an abelian category $\mathcal{A}$ with enough injectives, we obtain **right** derived functors:

$$R^iG(A) = \ H_i(G \circ I_\bullet \text{ and } R^iG(f : A \to A')$$

where $I_\bullet$ is an *injective* resolution of A, via **2 Proposition** applied with arrows reversed and injectives in place of projectives. Then every short-exact sequence:

$$0 \to A \to A' \to A'' \to 0$$

induces a long exact sequence of objects of $\mathcal{B}$:

$$0 \to G(A) \to G(A') \to G(A'') \to R^1G(A) \to R^1G(A') \to R^1G(A'') \to \cdots$$

Example. The right-derived functors of the left-exact $F_M = \mathrm{Hom}_R(M, \cdot)$ are:

$$\mathrm{Ext}^i_R(M, N) := R^iF_M(N), \text{ the Ext modules}$$

Thus, for example, letting $M = N''$, we have a long exact sequence:

$$0 \to \mathrm{Hom}(N'', N) \stackrel{f_*}{\to} \mathrm{Hom}(N'', N') \stackrel{g_*}{\to} \mathrm{Hom}(N'', N'') \stackrel{\delta^1}{\to} \mathrm{Ext}^1(N'', N) \to \cdots$$

associated to any short exact sequence of the form

$$(*)\ 0 \to N \to N' \to N'' \to 0$$

and the *extension class* $\epsilon(*) := \delta(1_{N''}) \in \mathrm{Ext}^1(N'', N)$ of the sequence is zero if and only if $1_{N''}$ is in the image of g_*, if and only if the sequence $(*)$ splits.

Interestingly, there is a converse to this. Given $\epsilon \in \mathrm{Ext}^1(N'', N)$, we can fashion a short exact sequence $(*)$ (in particular, constructing the module N' in the middle) with $\epsilon(*) = \epsilon$. Starting with an injective resolution:

$$0 \to N \stackrel{d_0}{\to} I_0 \stackrel{d_1}{\to} I_1 \stackrel{d_2}{\to} I_2 \to \cdots \text{ and}$$

we have, by definition, that ϵ is an element of the middle homology of:

$$\mathrm{Hom}(N'', I_0) \stackrel{d_{1*}}{\to} \mathrm{Hom}(N'', I_1) \stackrel{d_{2*}}{\to} \mathrm{Hom}(N'', I_2)$$

i.e. $\epsilon \in \mathrm{Hom}(N'', \ker(d_2)) = \mathrm{Hom}(N'', \mathrm{im}(d_1))$ (modulo the image of d_{1*}).

Now we add an injective resolution of N'' to the mix:

$$0 \to N'' \stackrel{d_0{''}}{\to} I_0{''} \stackrel{d_1{''}}{\to} I_1{''} \stackrel{d_2{''}}{\to} I_2{''} \to \cdots$$

Then, using the injectivity of I_1, we obtain $f : I_0{''} \to I_1$:

$$\begin{array}{ccc} I_1 & & \\ \uparrow \epsilon & \nwarrow f & \\ N'' & \stackrel{d_0{''}}{\to} & I_0{''} \end{array}$$

which we use to define a homomorphism:

$$\Phi = \left[\begin{array}{cc} d_1 & f \\ 0 & d_1{''} \end{array} \right] : I_0 \oplus I_0{''} \to I_1 \oplus I_1{''}$$

and then we obtain a commuting diagram:

$$\begin{array}{ccccccccc} 0 & \to & I_1 & \to & I_1 \oplus I_1'' & \to & I_1'' & \to & 0 \\ & & \uparrow d_1 & & \uparrow \Phi & & \uparrow d_1'' & & \\ 0 & \to & I_0 & \to & I_0 \oplus I_0'' & \to & I_0'' & \to & 0 \end{array}$$

with sequence (from the snake lemma):

$$0 \to N \to \ker(\Phi) \to N'' \xrightarrow{\delta} \mathrm{coker}(d_1) \to \mathrm{coker}(\Phi) \to \mathrm{coker}(d_1'') \to 0$$

But if $i_1, j_1 \in I_1$ and $(i_1, 0) - (j_1, 0) = 0$ as an element of $\mathrm{coker}(\Phi)$, then

$(i_1, 0) - (j_1, 0) = (i_1 - j_1, 0) = \Phi(i_0, i_0'') = (d_1(i_0) + f(i_0''), d_1''(i_0''))$ for some (i_0, i_0'') and then it follows that $d_1''(i_0'') = 0$, so $i_0'' = d_0''(n'')$ for some $n'' \in N''$ and also that $f(i_0'') = \epsilon(n'') \in \ker(d_2) = \mathrm{im}(d_1)$, so $i_1 - j_1 = d_1(i_0) + f(i_0'')$ is in the image of d_1 and $i_1 - j_1 = 0$ as an element of $\mathrm{coker}(d_1)$. All this is to say that the map following δ is injective, and so by exactness δ is the zero map! The truncated sequence:

$$(*)\ 0 \to N \to N' = \ker(\Phi) \to N'' \xrightarrow{\delta} 0$$

is the desired short exact sequence with $\epsilon(*) = \epsilon$. □

Problem. It is difficult to work with injective resolutions.

For the Ext functors there is a convenient fix, which we give without proof.

Theorem 4. Instead of computing $\mathrm{Ext}^i(M, N)$ as

$$H_i(\mathrm{Hom}(M, I_\bullet))$$

for an injective resolution $I_\bullet$ of N, we may instead compute it as:

$$H_i(\mathrm{Hom}(P_\bullet, N))$$

for a *projective* resolution of M (and the contravariant functor $F^N = \mathrm{Hom}(\bullet, N)$).

Remark. The same drill as for the 3 Theorems allow one to conclude that F^N has right derived functors, computed as $H_i(\mathrm{Hom}(P_\bullet, N))$. The surprising part of the Theorem is that this yields the *same* modules $\mathrm{Ext}^i(M, N)$.

Examples. The following sequence of abelian groups is clearly not split:

$$(*)\ 0 \to \mathbb{Z} \xrightarrow{(2,1)} \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \xrightarrow{1+4} \mathbb{Z}/6\mathbb{Z} \to 0$$

and so determines a nonzero class $\epsilon(*) \in \mathrm{Ext}^1(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z})$. We may compute this via:

$$0 \to \mathbb{Z} \xrightarrow{6} \mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \to 0$$

which we hit with the functor $F^{\mathbb{Z}}$ to get:

$$\mathbb{Z} = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{6^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$$

from which we conclude that $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) = \mathbb{Z}/6\mathbb{Z}$.

We know that the zero extension class gives the split sequence, but:

Question. Which extension class(es) give:

$$(*)\ 0 \to \mathbb{Z} \xrightarrow{(2,1)} \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \xrightarrow{1+4} \mathbb{Z}/6\mathbb{Z} \to 0?$$

and which extension class(es) give: $(**)\ 0 \to \mathbb{Z} \xrightarrow{6} \mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \to 0$?

and are we missing any other short exact sequences?

Popup Ad. Let X be a topological space, and consider the category $\mathcal{X}$ with:

The objects of $\mathcal{X}$ are the open subsets U of X.

The morphisms of $\mathcal{X}$ are the inclusions $U \subseteq V$.

A *contravariant* functor $\mathcal{A} : \mathcal{X} \to \mathcal{A}b$ (to the category of abelian groups) is a *presheaf* of abelian groups on X. In other words, a presheaf $\mathcal{F}$ consists of:

(i) An abelian group $\mathcal{A}(U)$ attached to each open subset, and

(ii) Restriction maps $\rho_{V,U} : \mathcal{A}(V) \to \mathcal{A}(U)$ attached to each $U \subseteq V$ such that:

- $\rho_{U,U} = 1_{\mathcal{A}(U)}$ and:
- $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U} : \mathcal{A}(W) \to \mathcal{A}(U)$ whenever $U \subseteq V \subseteq W$.

Example. The *constant* presheaf A (for a fixed abelian group A) is defined by:

$$A(\emptyset) = 0,\ A(U) = A \text{ and } \rho_{V,U} = 1_A \text{ for all } U \neq \emptyset$$

Rather amazingly, this is an interesting presheaf. It is associated to:

The *locally constant* sheaf A^+, defined by:

$A^+(U) = \{\text{continuous maps } f : U \to A \text{ for the discrete topology on } A\}$.

$A^+(U \subseteq V)$ is the restriction of continuous functions $f : V \to A$ to $f|_U : U \to A$.

Note that if U is connected, then $A(U) = A^+(U)$, since the continuous maps from a connected set to a set with the discrete topology are the constant maps! But if U has n connected components, then $A^+(U) = A^n$ and the restriction maps to each connected component are the projections.

There is a lot to say about this, but suffice it for the purposes of this teaser to say that there is a category of sheaves of abelian groups on the fixed topological space X with enough injectives, and that the covariant *global section functor*

$$\Gamma : \mathcal{A} \to \text{Ab};\ \mathcal{A} \mapsto \mathcal{A}(X)$$

is left-exact, which then defines right derived functors of the global section functor, which are the *cohomology* groups:

$$H^i(X, A) := R^i\Gamma(X, A)$$

These may be computed by taking a “good open cover” of X, and are basically dual to the singular cohomology of X (when $A = \mathbb{Z}$) that we discussed in an earlier popup topological ad.