Categories, Symmetry and Manifolds

Math 4800, Fall 2020

5. Groups. The symmetries of an object X (in a given category) form a group with composition as the group operation. It is instructive to consider the **category** of groups, in order to define the *action* of a group G on an object X as a morphism from G to the group of symmetries of X. An action of G on a vector space V is called a *representation* of G. There is a *conjugation* action of a group G on itself that breaks the group up into conjugacy classes. We revisit some of the finite groups we have encountered (e.g. C_n, D_{2n}, A_n, S_n) from this point of view and also begin to explore the circle and the special unitary group SU(2).

Definition 5.1. A **group** $(G, \cdot, 1)$ is a set G with an associative multiplication:

$$\cdot : G \times G \to G$$

and an element $1 \in G$ that is a two-sided identity: $1 \cdot g = g = g \cdot 1$ for all $g \in G$, such that each $g \in G$ has a (unique) two-sided inverse g^{-1} with $g^{-1}g = 1 = gg^{-1}$.

Examples. (a) Abelian groups have the additional commutative property, which can be captured by the *relation*:

$$gh = hg \text{ or equivalently } ghg^{-1}h^{-1} = 1 \text{ for all } g, h \in G$$

(b) The symmetries $(\text{Aut}(X), \circ, 1_X)$ of an object X of a category $\mathcal{C}$ are a group, with composition of symmetries as the group multiplication.

Definition 5.2. A subset $H \subset G$ is a **subgroup** if $1 \in H$ and H is closed under multiplication and inverses.

Examples. (a) The alternating subgroup $A_n \subset S_n = \text{Aut}([n])$.

(b) The Klein subgroup

$$K_4 = \{\text{id}_{[4]}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \subset A_4 \subset S_4$$

(c) If G is any group and $g \in G$, then:

$$\langle g \rangle = \{..., g^{-2}, g^{-1}, 1, g, g^2, ...\} \subset G \text{ is a cyclic subgroup of } G$$

This is either infinite cyclic or a cyclic group C_d. This is the smallest subgroup containing the element g. More generally, $\langle g_1, ..., g_n \rangle \subset G$ is the smallest subgroup containing $g_1, ..., g_n$. It consists of all "words" in letters $g_1, ..., g_n$ and $g_1^{-1}, ..., g_n^{-1}$.

(d) The group $O(n, \mathbb{R})$ of orthogonal transformations of $\mathbb{R}^n$ is a subgroup of the group of symmetries of the metric space $(\mathbb{R}, d)$ **and** a subgroup of the symmetries of the vector space $\mathbb{R}^n$. The group $T_{\mathbb{R}^n}$ of translations is another subgroup of the Euclidean group (but translations are not linear). We've seen subgroups:

$$C_n \subset D_{2n} \subset O(2, \mathbb{R}) \text{ and } A_4, S_4, A_5 \subset O(3, \mathbb{R})$$

defined as the symmetries of regular polygons and Platonic solids in §3.

Definition 5.2. (i) A **morphism** (or homomorphism) of groups is a function: $f : G \to H$ such that $f(1) = 1$ and $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$ for all $g_1, g_2 \in G$.

(ii) A morphism $\rho : G \to \text{Aut}(X)$ is an **action** of G on an object X.

This defines the category $\mathfrak{Gr}$ of groups. As in §2 and §4, we have:

Observation. If a morphism $f : G \to H$ of groups is a bijection, then $f^{-1} : H \to G$ is also a morphism of groups, so isomorphisms in $\mathfrak{Gr}$ are bijective morphisms.

Examples. (a) The sign of a permutation is a morphism $\mathrm{sgn} : S_n \to \{\pm 1\}$.

(b) The determinant of an $n \times n$ matrix is a morphism $\det : \mathrm{GL}(n, F) \to F^*$ from the **general linear group** of symmetries of F^n to the multiplicative group $(F^*, \cdot, 1) = \mathrm{GL}(1, F)$ of the field F.

(c) The permutation action of S_n on the vector space F^n is given by:

$$\rho_p : S_n \to \mathrm{GL}(n, F), \quad \rho_p(\sigma)(e_i) = e_{\sigma(i)}$$

(n = 2) The two permutations 1 and (1 2) map to:

$$\rho_p(1) = I_2 \text{ and } \rho_p(1\ 2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(n = 3) The six permutations $1, (1\ 2\ 3), (1\ 3\ 2), (1\ 2), (1\ 3), (2\ 3)$ map to:

$$\rho_p(1) = I_3,\ \rho_p(1\ 2\ 3) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ \rho_p(1\ 3\ 2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\ \text{etc}$$

(d) The cyclic group $C_d = \{1, g, ..., g^{d-1}, g^d = 1\}$ acts on the vector space $\mathbb{C}^1$ by:

$$\rho(g^k) = \text{ multiplication by } e^{2k\pi i/d} \in \mathrm{GL}(1, \mathbb{C}) = (\mathbb{C}^*, \cdot, 1)$$

If we instead view $\mathbb{C} = \mathbb{R}^2$, the action is by rotations:

$$\rho(g^k) = \begin{bmatrix} \cos(2k\pi/d) & -\sin(2k\pi/d) \\ \sin(2k\pi/d) & \cos(2k\pi/d) \end{bmatrix}$$

which we can think of as an action of C_d either on the vector space $\mathbb{R}^2$ or on the metric space $(\mathbb{R}^2, d)$, since orthogonal transformations are symmetries of both.

(e) Left multiplication determines an action of G on itself **as a set**:

$$\rho_l(g) = \text{left multiplication by } g, \text{ i.e. } \rho_l(g)(h) = gh$$

converting elements of G into permutations of the set G. It is an action because:

$$\rho_l(gg')(h) = (gg')h = g(g'h) = \rho_l(g) \circ \rho_l(g')(h)$$

so ρ_l converts group multiplication to composition of permutations.

For example if $G = S_3$, then $\rho_l : S_3 \to S_6$ maps elements of S_3 to permutations of the six elements of S_3. We can be explicit, ordering S_3 (with letters for clarity):

$$a = 1, b = (1\ 2\ 3), c = (1\ 3\ 2), d = (1\ 2), e = (1\ 3), f = (2\ 3)$$

and then $\rho_l(1) = \mathrm{id}, \rho_l(1\ 2\ 3) = (a\ b\ c)(d\ e\ f), \rho_l(1\ 2) = (a\ d)(b\ f)(c\ e)$, etc.

(f) In contrast, there is an action of G on itself **as a group** given by conjugation:

$$\rho_c(g) = \text{conjugation by } g, \text{ i.e. } \rho_c(g)(h) = ghg^{-1}$$

This is not just a permutation of the set G, but a **morphism** from G to itself, i.e. a symmetry of G in the category of groups, since:

$$\rho_c(1) = \mathrm{id} \text{ and } \rho_c(g)(hh') = g(hh')g^{-1} = (ghg^{-1})(gh'g^{-1}) = \rho_c(g)(h) \cdot \rho_c(g)(h')$$

It is an **action** since $\rho_c(gg')(h) = (gg')h(gg')^{-1} = g(g'hg'^{-1})g^{-1} = \rho_c(g) \circ \rho_c(g')(h)$.

Notice that the conjugation action of an **abelian** group on itself is trivial:

$$\rho_c(g)(h) = ghg^{-1} = hgg^{-1} = h \text{ for all } g \text{ and all } h$$

This is in contrast to the left multiplication action, which is never trivial.

If $f : G \to H$ is a morphism, then:

(a) The **image** $I = f(G) \subset H$ is a subgroup, and

(b) The **kernel** $K = f^{-1}(1) \subset G$ is a subgroup with the additional property:

$$(*) \text{ For all } \ k \in K \text{ and } g \in G,\ gkg^{-1} \in K$$

This property holds since $f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)f(g^{-1}) = f(1) = 1$.

Definition 5.3. Any subgroup $K \subset G$ with property $(*)$ above is called **normal** To distinguish normal subgroups from "ordinary" subgroups, one writes: $K \triangleleft G$.

Observation. If $H \subset G$ is a subgroup, then:

$$\rho_c(H) = \{ghg^{-1} \mid h \in H\}$$

is another subgroup of G. A subgroup $K \subset G$ is normal if $\rho_c(K) = K$.

Proposition 5.4. (a) The **left cosets** of a subgroup $H \subset G$ are the subsets:

$$gH = \{g \cdot h \mid h \in H\}$$

and the right cosets are:

$$Hg = \{h \cdot g \mid h \in H\}$$

The right (or left) cosets of H partition G into equivalence classes of the same cardinality as H, and we conclude that if G is a **finite** group, then $|H|$ divides $|G|$ for all subgroups (Lagrange's Theorem). In particular, the **order** of any element $g \in G$ (= the number of elements in $\langle g \rangle$) divides $|G|$ when G is a finite group.

(b) If $K \triangleleft G$ is a normal subgroup, then $gK = Kg$ for all g, and:

$$g_1K \cdot g_2K = (g_1g_2)K$$

is a well-defined multiplication on cosets, defining a **quotient** group G/K of cosets.

Examples. (a) Every subgroup of an abelian group is normal.

(b) The alternating subgroup $A_n \subset S_n$ is normal, since it is the kernel of the sign homomorphism (or you can verify it directly).

(c) The subgroup $S_n \subset S_{n+1}$ of permutations of $[n]$ inside permutations of $[n+1]$ is not normal, since, for example,

$$(n\ n+1)(1\ 2\ \cdots\ n)(n\ n+1) = (1\ 2\ \cdots\ n-1\ n+1) \notin S_n$$

(d) By direct computation, you can verify that $K_4 \subset S_4$ is a normal subgroup. Or else, one can directly find a homomorphism $f : S_4 \to S_3$ with kernel equal to K_4. One beautiful way to do this is to map a symmetry of the cube to the permutation group of the three "axles" of the cube (lines joining the midpoints of opposite sides).

Definition 5.5. A group G is **simple** if $\{1\}$ and G are its only normal subgroups.

Examples. (a) Among the abelian groups, in which every subgroup is normal, the only simple groups are the cyclic groups C_p of prime order.

(b) S_n for $n > 2$ are not simple, due to the normal subgroups $A_n \subset S_n$.

Theorem 5.6. The alternating groups A_n for $n \geq 5$ are simple.

(See an advanced algebra class for the proof of this.).

We next turn to the conjugacy classes of a group. This is a list of the "orbits" of the elements of G under the action of conjugation.

Definition 5.7. Elements $h, h' \in G$ are **conjugate** if:

$$\rho_c(g)(h) = ghg^{-1} = h' \text{ for some } g \in G$$

This induces an equivalence relation on G by:

$$h \sim h' \text{ if and only if } h \text{ and } h' \text{ are conjugate}$$

The equivalence classes are the **conjugacy classes** of G.

Remark. A subgroup of G is normal if and only if it is a union of conjugacy classes!

Finite Examples. (a) Cyclic groups. The conjugacy classes of C_n are singletons:

$$\{1\}, \{x\}, \{x^2\}, ..., \{x^{n-1}\}$$

as are the conjugacy classes of all abelian groups.

(b) Dihedral groups. From the relations $x^n = 1 = y^2$ and $yx = x^{-1}y$ on:

$$D_{2n} = \{1, x, x^2, ..., x^{n-1}, y, xy, \dots, x^{n-1}y\}$$

we get conjugacy classes:

$$\{1\}, \{x, x^{-1}\}, ..., \{x^k, x^{-k}\}, ..., \{y, yx^2, yx^4, ...\}, \{yx, yx^3, ..., \}$$

but this needs to be taken with a grain of salt. For example, for small n:

$$D_4 : \{1\}, \{x\}, \{y\}, \{yx\} \text{ since } x = x^{-1}$$

$$D_6 : \{1\}, \{x, x^2\}, \{y, yx^2, yx\} \text{ since } x^4 = x$$

$$D_8 : \{1\}, \{x, x^3\}, \{x^2\}, \{y, yx^2\}, \{yx, yx^3\}$$

$$D_{10} : \{1\}, \{x, x^4\}, \{x^2, x^3\}, \{y, yx^2, yx^4, yx, yx^3, yx^5\}$$

(c) Groups of permutations. If $\sigma : [n] \to [n]$ is a symmetry, then conjugating a symmetry in cycle notation by σ has the effect of replacing each entry i with $\sigma(i)$. In other words, if $\tau(i) = j$, then:

$$\sigma \circ \tau \circ \sigma^{-1}(\sigma(i)) = \sigma \circ \tau(i) = \sigma(j)$$

Thus, for example, if $\sigma = (1\ 2\ 3)$, then:

$$\sigma(1\ 3)\sigma^{-1} = (\sigma(1)\ \sigma(3)) = (2\ 1) = (1\ 2)$$

As a consequence, the **partitions** of $[n]$ are in bijection with the conjugacy classes.

$$S_2 : \{(1)(2)\}, \{(1\ 2)\}$$

$$S_3 : \ \{(1)(2)(3)\}, \{(1\ 2)(3), (1\ 3)(2), (1)(2\ 3)\}, \{(1\ 2\ 3), (1\ 3\ 2)\}$$

$$S_4 : \ \{(*)(*)(*)(*)\}, \{(*\ *)(*)(*)\}, \{(*\ *)(*\ *)\}, \{(*\ \ *\ \ *)(*)\}, \{(*\ \ *\ \ *\ *)\}$$

Infinite Examples. (a) Conjugating a rotation by a reflection gives:

$$\rho_{\theta_2/2} \circ \phi_{\theta_1} \circ \circ \rho_{\theta_2/2} = \rho_{\theta_2/2} \circ \rho_{(\theta_1+\theta_2)/2} = \phi_{-\theta_1}$$

and conjugating a reflection by a rotation gives:

$$\phi_{\theta_2}\rho_{\theta_1/2}\phi_{-\theta_2} = \rho_{(\theta_1/2)+\theta_2}$$

and the conjugacy classes of the *infinite dihedral group* $\mathrm{O}(2, \mathbb{R})$ are:

(i) All reflections are in the same conjugacy class $\mathrm{O}(2, \mathbb{R})^-$ (a circle!), and

(ii) Conjugacy classes of rotations are either $\{\phi_0\}, \{\phi_\pi\}$ or else $\{\phi_\theta, \phi_{-\theta}\}$.

(d) In the group $\mathrm{GL}(n,\mathbb{C})$, recall from §4 that if:

$$A_1 \sim A_2 \text{ then } \mathrm{ch}(A_1) = \mathrm{ch}(A_2)$$

and in particular, the **eigenvalues** of A_1 and A_2 are the same.

• A matrix A is semi-simple if a diagonal matrix is in the conjugacy class of A, and the location of the eigenvalues on the diagonal are permuted under conjugation by permutation matrices, so there is one semi-simple conjugacy class for every list of complex numbers (maybe with repetitions): $\{\lambda_1, ..., \lambda_n\}$.

• General matrices have one conjugacy class of every list of "Jordan blocks."

Unitary Groups. The standard Hermitian inner product on $\mathbb{C}^n$ is:

$$\langle \vec{z}, \vec{w} \rangle = \langle (z_1, ..., z_n), (w_1, ..., w_n) \rangle = \sum_{i=1}^{n} z_i \overline{w}_i \in \mathbb{C}$$

It is not bilinear, but rather "conjugate" bilinear:

$$\langle \vec{w}, \vec{z} \rangle = \overline{\langle \vec{z}, \vec{w} \rangle},\ \langle c\vec{z}, \vec{w} \rangle = c\langle \vec{z}, \vec{w} \rangle \text{ and } \langle \vec{z}, c\vec{w} \rangle = \overline{c}\langle \vec{z}, \vec{w} \rangle$$

It is positive definite, in the sense that:

$$\langle \vec{z}, \vec{z} \rangle = \sum_{i=1}^{n} z_i \overline{z}_i = \sum_{i=1}^{n} |z_i|^2$$

is real and strictly positive for all non-zero vectors $\vec{z} \in \mathbb{C}^n$.

Definition 5.8. A $\mathbb{C}$-linear map $f : \mathbb{C}^n \to \mathbb{C}^n$ is a **unitary** transformation if:

$$\langle f(\vec{z}), f(\vec{w}) \rangle = \langle \vec{z}, \vec{w} \rangle$$

for all vectors $\vec{z}, \vec{w} \in \mathbb{C}^n$.

This is the complex analogue of an orthogonal transformation. The unit sphere:

$$S^{2n-1} = \{\vec{u} = (s_1 + it_1, ..., s_n + it_n) \in \mathbb{C}^n \mid \langle \vec{u}, \vec{u} \rangle = \sum_{i=1}^{n} s_i^2 + t_i^2 = 1\}$$

is preserved under a unitary transformation, as is orthogonality:

$$\langle \vec{z}, \vec{w} \rangle = 0 \ \Rightarrow \ \langle f(\vec{z}), f(\vec{w}) \rangle = 0$$

Thus to specify a unitary transformation A (in matrix form), one specifies:

$$f(e_1) = \vec{u}_1, ..., f(e_n) = \vec{u}_n, \text{ pairwise orthogonal unit vectors in } S^{2n-1}$$

and then in this case,

$$A \cdot \overline{A^T} = I_n$$

from which it follows that $\det(A) \cdot \overline{\det(A)} = 1$ and so $\det(A) = e^{i\theta} \in \mathbb{C}$ for some θ.

Definition 5.9. (a) $U(n)$ is the group of unitary transformations of $\mathbb{C}^n$.

(b) $SU(n) \subset U(n)$ is the subgroup of unitary transformations of determinant 1. This is called the **special** unitary group.

Example. The Unitary group $U(1)$ consists of the unit circle of rotations of $\mathbb{C}$.

Theorem 5.10. The special unitary group SU(2) is isomorphic to the group of **unit quaternions**, each giving the three-sphere S^3 the structure of a group.

Proof. The quaternions (or Hamiltonians) are the $\mathbb{R}$-vector space:

$$\mathbb{H} = \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$$

with an associative (but not commutative) multiplication defined by linearity and:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{k} = -\mathbf{j} \cdot \mathbf{i}, \quad \mathbf{j} \cdot \mathbf{k} = \mathbf{i} = -\mathbf{k} \cdot \mathbf{j}, \quad \mathbf{k} \cdot \mathbf{i} = \mathbf{j} = -\mathbf{i} \cdot \mathbf{k}$$

It follows that:

$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(a - b\mathbf{i} - b\mathbf{j} - c\mathbf{k}) = a^2 + b^2 + c^2 + d^2$$

This is interpreted as multiplication by the quaternionic conjugate and

$$(a + b\vec{i} + c\vec{j} + d\vec{k})^{-1} = \frac{\overline{(a + b\vec{i} + c\vec{j} + d\vec{k})}}{a^2 + b^2 + c^2 + d^2}$$

making the quaternions into a *division algebra* (with all the properties of a field other than commutativity of multiplication). The unit quaternions:

$$S^3 = \{\vec{u} \in \mathbb{H} \mid \vec{u} \cdot \overline{\vec{u}} = 1\}$$

are thus a group with quaternion multiplication.

An element of $\mathrm{SU}(2, \mathbb{C})$ is a 2×2 matrix of orthonormal complex vectors:

$$A = \begin{bmatrix} s_1 + it_1 & u_1 + iv_1 \\ s_2 + it_2 & u_2 + iv_2 \end{bmatrix}$$

of determinant one. Up to multiplication by a (complex) scalar λ,

$$(s_1 + it_1, s_2 + it_2)^\perp = (s_2 - it_2, -(s_1 - it_1))$$

and with the condition $s_1^2 + t_1^2 + s_2^2 + t_2^2 = 1$ and $\det(A) = 1$, we get $\lambda = -1$ and:

$$A = \begin{bmatrix} s_1 + it_1 & -s_2 + it_2 \\ s_2 + it_2 & s_1 - it_1 \end{bmatrix} = s_1 I_2 + t_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + s_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

and one checks that the morphism defined by:

$$I_2 \mapsto 1, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \mapsto \mathbf{i}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mapsto \mathbf{j}, \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \mapsto \mathbf{k}$$

determines the desired isomorphism of groups. □

Proposition 5.11. The unit sphere $S^2 \subset \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ "equator" is a single conjugacy class of $\mathrm{SU}(2) = S^3 \subset \mathbb{H}$ and the conjugation action of SU(2) on S^2 is by (special) orthogonal transformations.

Proof. Group conjugation by $a + \vec{v} \in \mathrm{SU}(2)$ is a linear transformation, since quaternionic multiplication is bilinear. Notice that the quaternion conjugate $\overline{a + \vec{v}} = a - \vec{v}$ is the quaternion inverse, since $a^2 + |\vec{v}|^2 = 1$ by assumption. Quaternion multiplication is given in terms of the dot an cross product of vectors in $\mathbb{R}^3$ by:

$$(a + \vec{v})(b + \vec{w}) = (ab - \vec{v} \cdot \vec{w}) + (a\vec{w} + b\vec{v} + \vec{v} \times \vec{w})$$

Now suppose $\vec{u} \in S^2$, and $a + l\vec{u} \in \mathrm{SU}(2)$. Then $\vec{u}(a - l\vec{u}) = l + a\vec{u}$ and:

$$(a + l\vec{u})\vec{u}(a - l\vec{u}) = (a + l\vec{u})(l + a\vec{u}) = al + a^2\vec{u} + l^2\vec{u} - al = \vec{u}$$

so $\vec{u}$ is a fixed vector for this linear transformation.

Now suppose that $\vec{v} \in S^2$ and let $\vec{w} = \vec{v} \times \vec{u}$. Then $\vec{w} \in S^2$ and $\vec{w}$ perpendicular to both $\vec{v}$ and $\vec{u}$. Then $\vec{v}(a - l\vec{u}) = a\vec{v} - l(\vec{w})$ and:

$$(a+l\vec{u})\vec{v}(a-l\vec{u}) = (a+l\vec{u})(a\vec{v}-l\vec{w}) = a^2\vec{v}-al\vec{w}+al(-\vec{w})-l^2\vec{v} = (a^2-l^2)\vec{v}-(2al)\vec{w}$$

and similarly, $(a + l\vec{u})\vec{w}(a - l\vec{u}) = (2al)\vec{v} + (a^2 - l^2)\vec{w}$.

But then in terms of the basis $\{\vec{u}, \vec{v}, \vec{w}\}$ for $\mathbb{R}^3$, conjugation by $(a + l\vec{u})$ is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \text{ with } \cos(\theta) = (a^2 - l^2) \text{ and } \sin(\theta) = -2al$$

i.e. it is the rotation by θ about the fixed vector $\vec{u}$, and an element of SO(3)!

From this, we get the famous "double cover" surjective group homomorphism:

$$\phi : \mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R}) \text{ with kernel } \phi^{-1}(I_3) = \pm 1$$

Since the action of $\mathrm{SO}(3, \mathbb{R})$ is *transitive*, i.e. for each pair of vectors $\vec{u}_1, \vec{u}_2 \in S^2$, there is an orthogonal transformation taking $\vec{u}_1$ to $\vec{u}_2$, it follows that S^2 is a **single** conjugacy class for the action of SU(2). $\square$

Remark. All special orthogonal groups $\mathrm{SO}(n, \mathbb{R})$ for $n \geq 3$ have canonical double covers by the so-called "spin" groups. Only when $n = 3$, however, is the spin group also a (special) unitary group.

Assignment 5.

1. (a) Find a cyclic subgroup of S_5 with 6 elements. Is it a normal subgroup?

(b) Is there a subgroup of S_4 with 8 elements? If so, how many are there?

2. Show that if G is a group with $2n$ elements and $H \subset G$ is a subgroup with n elements, then H is necessarily a normal subgroup of G.

3. (a) Find all the conjugacy classes of A_4.

(b) Find all the conjugacy classes A_5. Hint: They are not the same as the conjugacy classes of S_5 that happen to be in A_5, but it is a good start to find these, since the conjugacy classes of A_5 are subsets of them.

4. Identify the left cosets of the subgroups $S_n \subset S_{n+1}$ with the sets:

$$L_i = \{\sigma \in S_{n+1} \mid \sigma(n + 1) = i\}$$

and the right cosets with the sets:

$$R_i = \{\sigma \in S_{n+1} \mid \sigma(i) = n + 1\}$$

and in particular show that these are not the same!

5. When $\rho : G \to \mathrm{Aut}(X)$ is the action of a group G on a set X, the **stabilizer** of an element $x \in X$ is $G_x := \{g \in G \mid \rho(g)(x) = x\} \subset G$.

(a) Prove that G_x is a subgroup of G.

The G-orbit of $x \in X$ is $Gx := \{y \in X \text{ such that } \rho(h)(x) = y \text{ for some } h \in G\}$.

(b) Find a bijection between the left cosets hG_x of G_x and $Gx \subset X$.

(c) If $y \in Gx$, prove that G_x and G_y are conjugate subgroups of G.

(d) Find the stabilizer of $(0, 0, 1)$ for the action of $\mathrm{SO}(3, \mathbb{R})$ on the sphere S^2.

(e) Find the stabilizer of $(0, 0, 1)$ for the action of SU(2) on the sphere S^2.