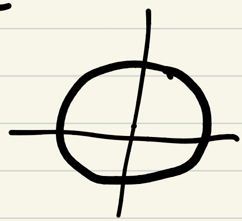
4800-15

Thm: SU(2) is isomorphic to the group of unit quaternions.



Pf: Unit length quaternions

$$S^3 \subset \mathbb{H} = \mathbb{R} + \mathbb{R}\vec{i} + \mathbb{R}\vec{j} + \mathbb{R}\vec{k}$$

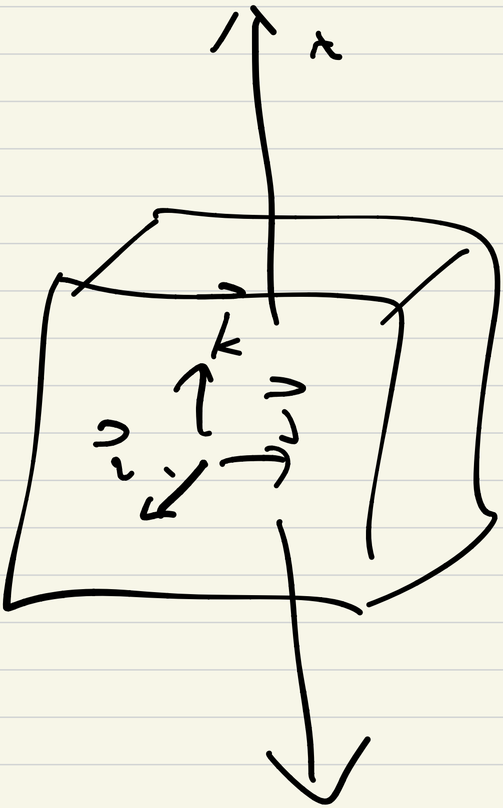
$$S^3 = \{ a + \vec{v} \mid a^2 + |\vec{v}|^2 = 1 \}$$

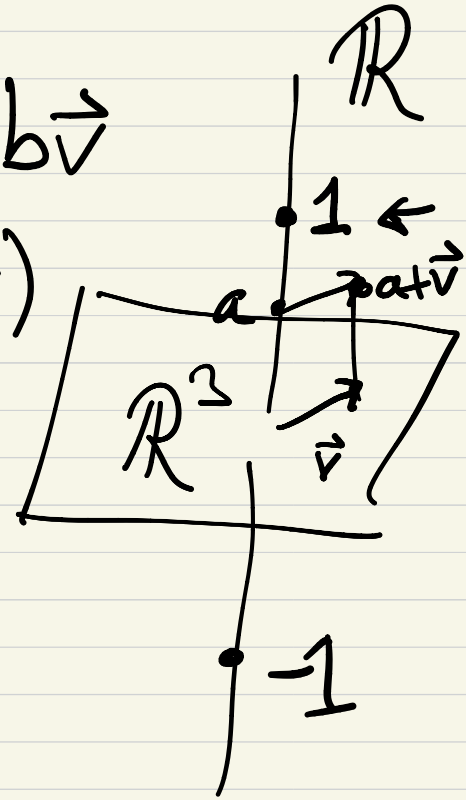
$$(a+\vec{v})\overline{(a+\vec{v})} = 1$$

$$\left(\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \right)$$

$$(\overset{\mathbb{R}}{a}+\overset{\mathbb{R}^3}{\vec{v}})\cdot(\overset{\mathbb{R}}{b}+\overset{\mathbb{R}^3}{\vec{w}})$$

$$=\left(ab+a\vec{w}+b\vec{v}-\vec{v}\cdot\vec{w}+\vec{v}\times\vec{w}\right)$$





$\underline{\underline{\mathbb{H}}}$

$$\vec{i}^2 = -1$$
$$\vec{j}^2 = -1$$
$$\vec{k}^2 = -1$$

$$\leftrightarrow$$

$$\vec{i} \times \vec{j} = \vec{k}$$
$$\vec{j} \times \vec{k} = \vec{i}$$
$$\vec{k} \times \vec{i} = \vec{j}$$

$\times$

$$(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) \cdot (w_1\vec{i} + w_2\vec{j} + w_3\vec{k})$$

$$= -v_1w_1 - v_2w_2 - v_3w_3$$

$$+(\vec{v} \times \vec{w})$$

$$= -\vec{v} \cdot \vec{w} + \vec{v} \times \vec{w}$$

SU(2) ; column vectors: unit + orthogonal

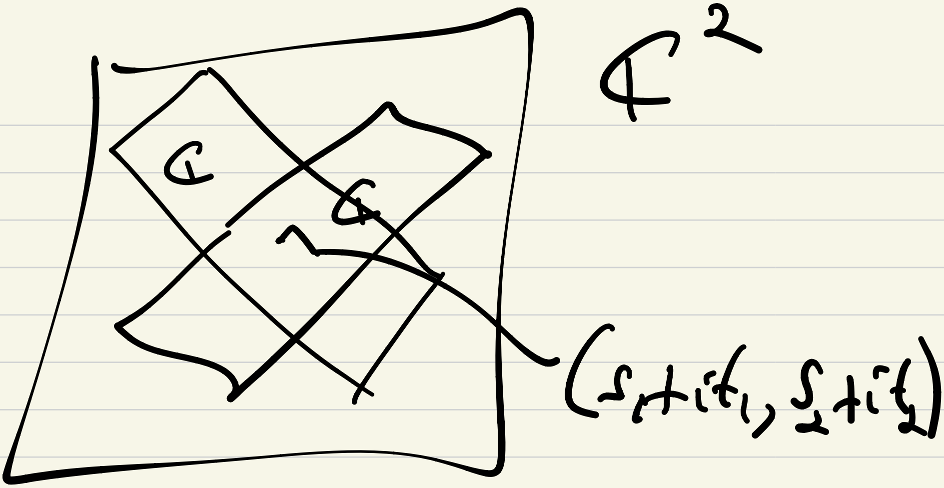
U(2) $\begin{pmatrix} s_1+it_1 & u_1+iv_1 \\ s_2+it_2 & u_2+iv_2 \end{pmatrix}$

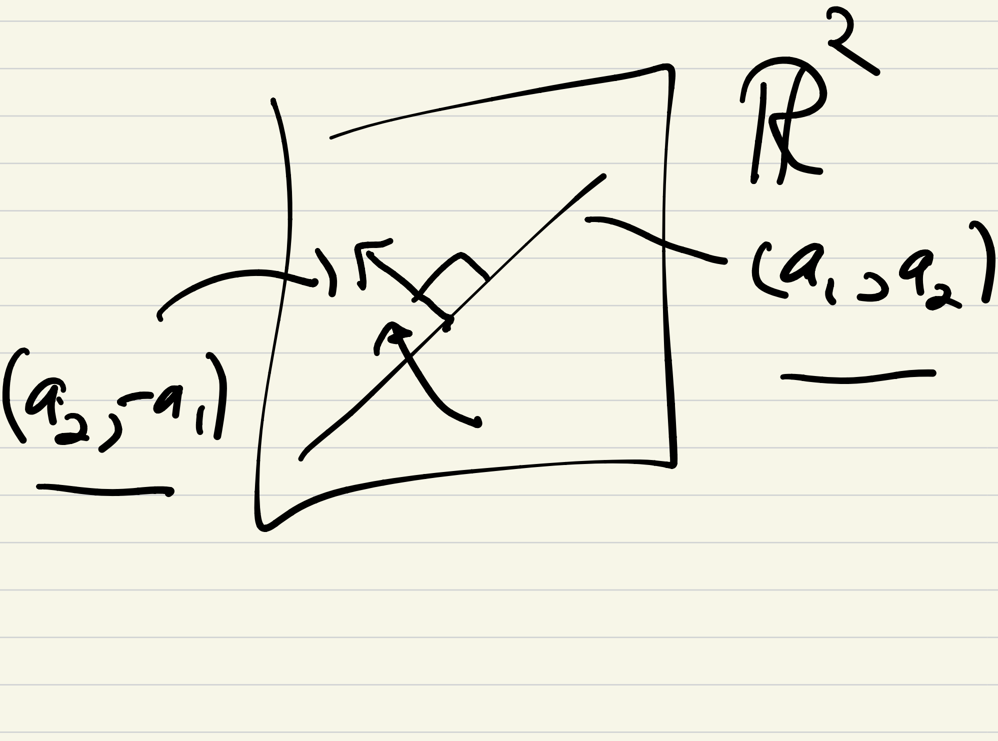
$s_1^2+t_1^2+s_2^2+t_2^2=1$

||

2×2 matrix of cx. numbers

with unit, orthogonal columns.


$\mathbb{C}^2$
$\mathbb{C}$
$\mathbb{C}$
(s_1+it_1, s_2+it_2)


$\mathbb{R}^2$
(a_1, a_2)
$(a_2, -a_1)$

$$\overset{z_1}{(s_1+it_1}, \overset{z_2}{s_2+it_2)}$$

$$\lambda\,(\overset{\overline{z_2}}{s_2-it_2}, \overset{-\overline{z_1}}{-(s_1-it_1)})$$

$$\langle (z_1, z_2), (\overline{z_2}, -\overline{z_1}) \rangle$$

$$= z_1 z_2 - z_2 z_1 = 0$$

$$\det\begin{pmatrix} z_1 & \lambda\overline{z_2} \\ z_2 & -\lambda\overline{z_1} \end{pmatrix}$$

$$= -\lambda z_1\overline{z_1} - \lambda z_2\overline{z_2} = -\lambda \quad (\lambda = -1)$$

$$SU(2) \ni \begin{pmatrix} s_1 + it_1 & -s_2 + it_2 \\ s_2 + it_2 & s_1 - it_1 \end{pmatrix}$$

$\mathbb{R}$

$$\boxed{s_1^2 + s_2^2 + t_1^2 + t_2^2 = 1}$$

Break this up

$$\left(s_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + s_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - t_2 \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right)$$

The isomorphism is:

$$f : SU(2) \xrightarrow{\sim} \text{unit } Q_s$$

$$f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$$

$$f\left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right) = \vec{i}$$

$$f\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \vec{j}$$

$$f\left(\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}\right) = \vec{k}$$

This is a group homomorphism.

$$\bar{i}^2 = -1$$

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\bar{j}^2 = -1$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\bar{k}^2 = -1$$

$$\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\vec{i} \cdot \vec{j} = \vec{k}$$

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$\vec{j} \cdot \vec{k} = \vec{i}$$ ✓

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\vec{j} \cdot \vec{i} = -\vec{k}$$ ✓

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

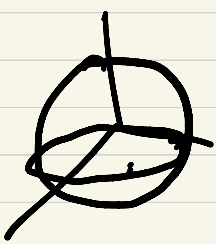
$$1 \longleftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

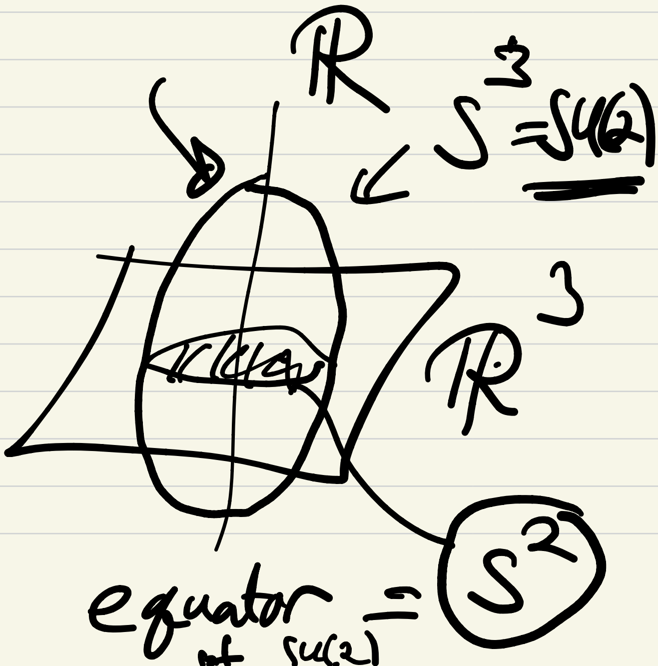
$$\vec{i} \longleftrightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\vec{j} \longleftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\vec{k} \longleftrightarrow \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$SU(2)$





Claim:

$$\underline{S^2} = \underline{su(2) \cap \mathbb{R}^3}$$

is a conjugacy class of $SU(2)$, and conjugatory by $a+\vec{v}$ is a function

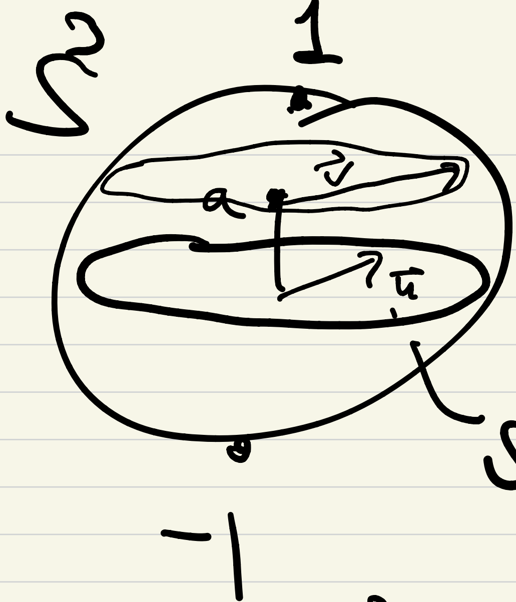
$$\underline{(a+\vec{v})} \cdot \underline{S^2} \cdot \underline{(a+\vec{v})^{-1}} \quad \rho_{conj}(a+\vec{v})$$

is an orthogonal transformation of S^2.

$$SU(2) = \left\{ (a, b, c, d) \,\middle|\, a^2 + b^2 + c^2 + d^2 = 1 \right\}$$

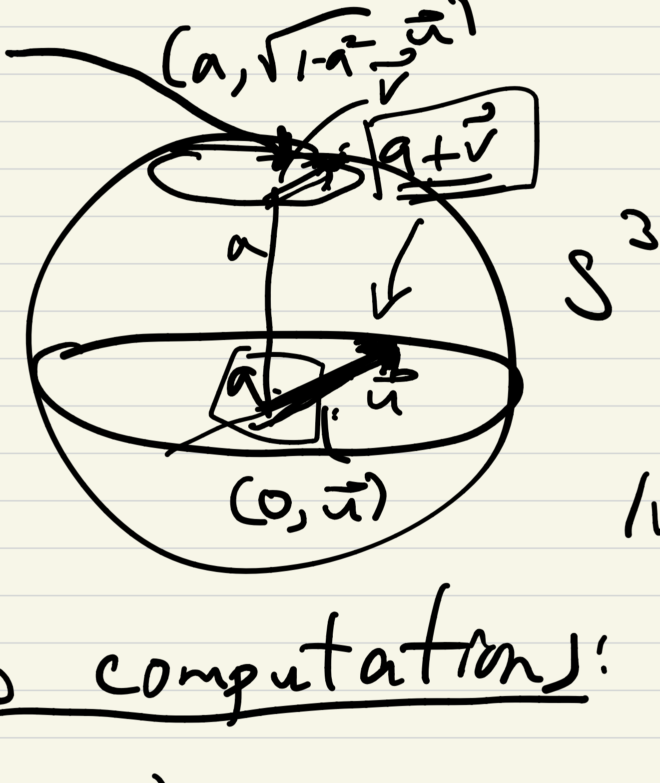
$$SU(2) \cap \underset{\sim}{\mathbb{R}}^3$$

$$= \left\{ (0, \underline{b, c, d}) \,\middle|\, b^2 + c^2 + d^2 = 1 \right\}.$$



$SU(2)$

$S^2 = (0, \vec{u})$



unit vector.

$$\boxed{\vec{v} = l\vec{u}}$$

$$|\vec{v}|^2 = 1 = a^2 + l^2$$

$$l = \sqrt{1-a^2}$$

3 computations:

$$(a + l\vec{u}) \cdot \vec{u} \cdot (a + l\vec{u})^{-1} = \vec{u}$$

$(a + l\vec{u})$

Let $\vec{v} \perp \vec{u}$ in S^2

Then:

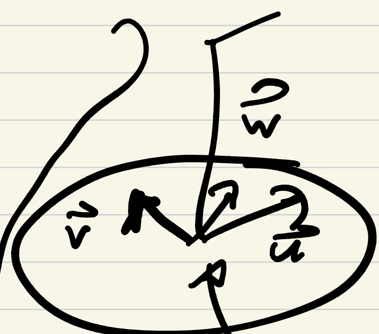
(2) $(a+l\vec{u})\vec{v}(a+l\vec{u})^{-1}$

$(\vec{w} = \vec{v} \times \vec{u})$

$$= \underline{(a^2-l^2)}\vec{v} + \underline{(-2al)}\,\vec{w}$$

(3) $(a+l\vec{u})\vec{w}(a+l\vec{u})^{-1}$

$$= 2al\vec{v} + (a^2-l^2)\vec{w}$$



S^2

(basis for $\mathbb{R}^3$)

$\vec{u}, \vec{v}, \vec{w}$

Conjugation by $a+l\vec{u}$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a^2-l^2 & 2al \\ 0 & -2al & a^2-l^2 \end{bmatrix}$$

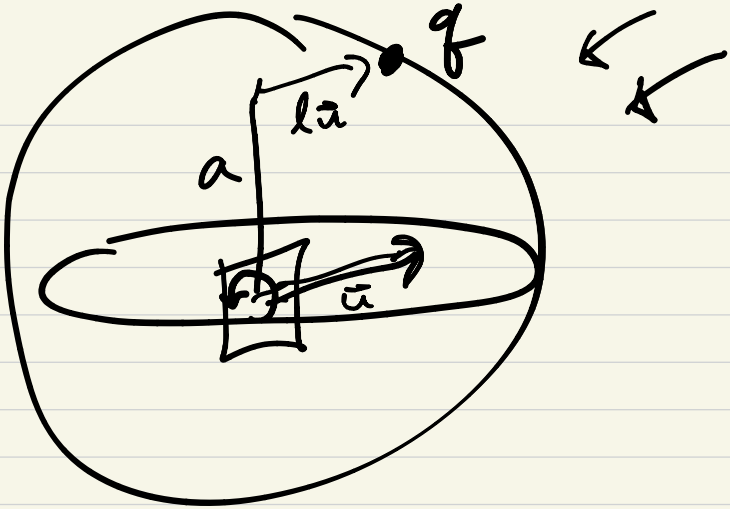
$\sin(\theta)$

$\cos(\theta)$

$\cos(\theta)$ $\sin(\theta)$

orthogonal!

$\vec{u}, \vec{v}, \vec{w}$



1

$$(a + l\vec{u})\left(\vec{u}(a - l\vec{u})\right)$$

$$= (a + l\vec{u})(l + a\vec{u})$$

$$= (\cancel{al} + a^2\vec{u} + l^2\vec{u} - \cancel{al})$$

$$= (a^2 + l^2)\vec{u} = \vec{u}$$

$$(\vec{v}\cdot\vec{u}=0) \qquad \vec{v}\perp\vec{u}=0$$

2

$$(a+l\vec{u})\left(\vec{v}(a-l\vec{u})\right)$$

$$=(a+l\vec{u})(\underline{a\vec{v}-l\vec{w}})$$

$$\vec{w}=(\vec{v}\times\vec{u})$$

$$=(a^2\vec{v}-al\vec{w}+al\underbrace{\vec{u}\times\vec{v}}_{=-\vec{w}}$$

$$-l^2\underbrace{\vec{u}\times\vec{w}}_{\vec{u}\times(\vec{v}\times\vec{u})=\vec{v}}$$

$$=\underline{((a^2-l^2)\vec{v}-2al\vec{w})}$$

✓

③ Same as ②

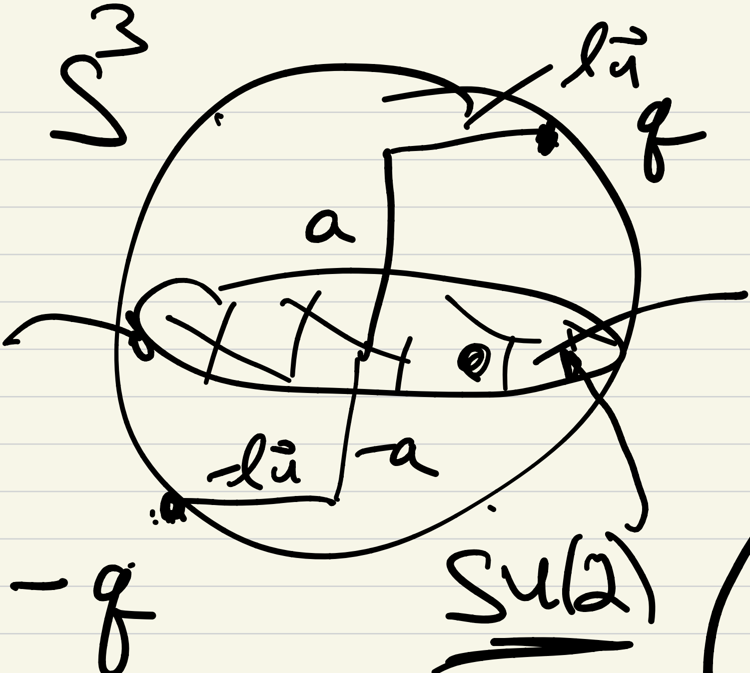
---

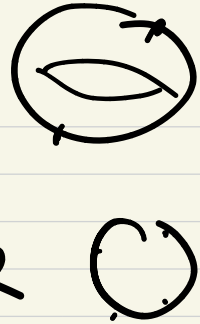
conjugation of S^2 = special orthogonal

$$SU(2) \xrightarrow{2:1} SO(3,\mathbb{R})$$

$q\vec{u}q^{-1}$
"
$(-q)\vec{u}(-q)^{-1}$

special orth. transf (symmetries of S^2)





$\mathbb{RP}^3$

$\|$

$$\phi : S^3 \xrightarrow{2:1} SO(3,\mathbb{R})$$

$\|$

$\boxed{SU(2)}$

$$\phi(q) = \phi(-q)$$

Munkres: Analysis

Spivak: Calculus on Manifolds

- Rudin: Analysis
- Warren: Introduction to Manifolds

Applied Linear Algebra and Matrix Analysis

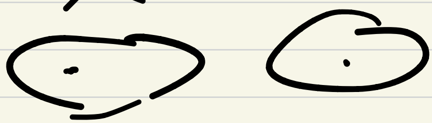
Introduction to Manifolds

Cartesian

$$\begin{tikzcd}
Z \arrow[rr, dashed, "{(f,g)}"] \arrow[dr, "f"'] \arrow[drrr, "g"] & & X \times Y \arrow[dl] \arrow[dr] & \\
& X & & Y
\end{tikzcd}$$

$\cdot$ $\downarrow$

$$X \overset{co}{\longleftrightarrow} Y$$



Disjoint Union

$$\begin{tikzcd}
Z & & X \sqcup Y \arrow[ll, dashed, "f \sqcup g"'] & \\
& X \arrow[ul, "f"] \arrow[ur] & & Y \arrow[ulll, "g"'] \arrow[ul]
\end{tikzcd}$$

$+$

Tensors: **Multilinear maps**

of

Vector Spaces

$$V \times V \xrightarrow{\cdot} \mathbb{R}$$

$$(\vec{v}_1 + \vec{v}_2) \cdot \vec{w} = \vec{v}_1 \cdot \vec{w} + \vec{v}_2 \cdot \vec{w}$$

$$\underline{V} \times \dots \times \underline{V} \xrightarrow{\text{multilinear}} \mathbb{R}$$

$$f(\vec{v}_1, \dots, \vec{v}_n)$$

(linear in each component)

$$V \xrightarrow{\text{linear}} \mathbb{R}$$

$e_1, \ldots, e_n$ basis for V

$x_1, \ldots, x_n$

$$\underline{x_i}(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$x_i(a_1, \ldots, a_n) = a_i$$

Sym.

$x_i \otimes x_j + x_j \otimes x_i$

$$V \times V \longrightarrow \mathbb{R}$$

$x_i \otimes x_j - x_j \otimes x_i$

alto

$$\boxed{x_i \otimes x_j}\big((a_1, \ldots, a_n), (b_1, \ldots, b_n)\big) = a_i b_j$$