4800 – 13

Groups :

* $(G, \cdot, 1)$ — objects
* $f : G \to H$ — morphisms

$f(1) = 1, \; f(g_1 g_2) = f(g_1) \cdot f(g_2)$

$f(g^{-1}) = (f(g))^{-1}.$

Gr category of groups

$\cup$

Ab " " abelian gps

Example: The group $(\mathrm{Aut}(X), \circ, \mathrm{id}_X)$ of symmetries of an object X in a category $\mathcal{C}$.

An action of G on X is a morphism

$$\rho : G \longrightarrow \mathrm{Aut}(X)$$

$$\rho(\underline{g}) : X \longrightarrow X$$

Example: Action of S_n on F^n:

$$\rho : S_n \longrightarrow GL(n, F) \text{ (invertible)}$$

$$GL(n,F) = \mathrm{Aut}(F^n)$$

$\sigma : [n] \to [n]$

$\rho(\sigma) = A$ with

$$A \cdot e_i = e_{\sigma(i)}$$

$$\rho : S_2 \longrightarrow GL(2, F)$$

$$\rho((1)(2))(e_1) = e_1$$

$$\rho((1)(2))(e_2) = e_2$$

$$I_2 = \rho(id)$$

$$\rho(id) = I_2$$

$$\rho(1\ 2)(e_1) = e_2$$

$$\rho(1\ 2)(e_2) = e_1$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \rho(1\ 2)$$

$$\rho : S_3 \longrightarrow GL(3, F)$$

$$\rho(id) = I_3$$

$$\rho(123) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \rho(132)$$

$$\rho((1\,2)(3)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Permutation matrices

$$* \quad \det(\rho(\sigma)) = \operatorname{sgn}(\sigma) \quad *$$

$\underline{C_d}$ acts $\underline{\text{on}}$ $\underline{\mathbb{C}^1}$ by:

$= \{1, x, x^2, \ldots, x^{d-1}, x^d = 1\}$

$$\rho(x) = e^{2\pi i/d}$$

$$\rho(x^k) = \left(e^{2\pi i/d}\right)^k$$

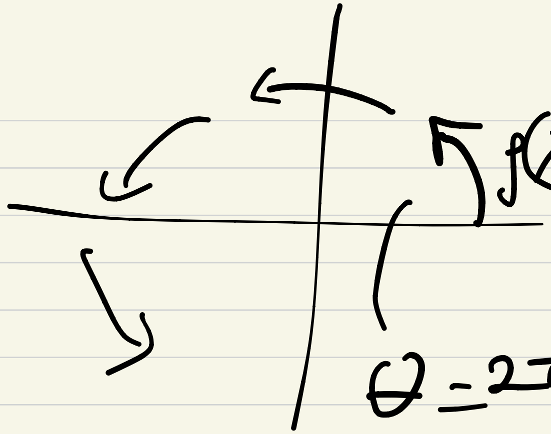
$$= e^{2k\pi i/d}$$

converts x^k into mult.

by $e^{2k\pi i/d}$ =

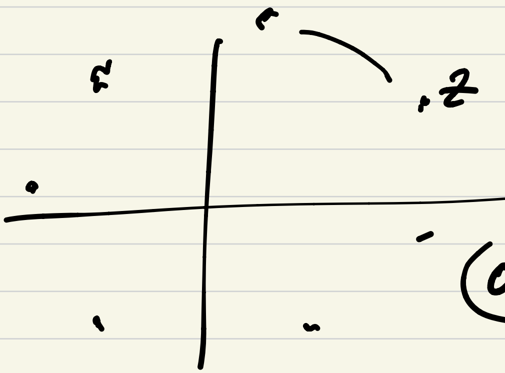
(rotation by $2\pi k/d$)

$\underline{\underline{C_d}}$



$f(x) = e^{2\pi i/d}$

$\theta = 2\pi/d$



(orbit of $\underline{z}$)

$$\rho : G \to \mathrm{Aut}(X) \supset G$$

$$\underline{g} \rightsquigarrow (\sigma : X \to X)$$

- A group G acts on <u>itself</u>:

(1) By left multiplication.

(2) By <u>conjugation</u>

<u>Left mult.</u>:

$$\mathcal{A}_\ell : G \longrightarrow \underset{\underline{\text{Sets}}}{\mathrm{Aut}}(G)$$

$$(\mathcal{A}_\ell(g))\, h = g \cdot h$$

Example: $G = S_3$

$\{1, (1\ 2\ 3), (1\ 3\ 2), (1\ 2), (1\ 3), (2\ 3)\}$

$\rho_\ell(1\ 2\ 3) = (a\ b\ c)(d\ e\ f)$

$1 \longrightarrow (1\ 2\ 3) \cdot 1 = (1\ 2\ 3)$

$(1\ 2\ 3) \longrightarrow (1\ 2\ 3)(1\ 2\ 3) = (1\ 3\ 2)$

$(1\ 3\ 2) \longrightarrow 1$

$(1\ 2) \longrightarrow (1\ 2\ 3)(1\ 2) = (1\ 3)$

$(1\ 3) \longrightarrow (2\ 3)$

$(2\ 3) \longrightarrow (1\ 2)$

$$\left\| \rho_\ell : S_3 \longrightarrow S_6 \right\|$$

$$\rho_\ell : G \longrightarrow S_{|G|}$$

$\underline{\rho_\ell}$ $\underline{\text{is}}$ $\underline{\text{an}}$ permutation $\underline{\text{action}}$:

$$\underline{\underline{\rho_\ell(g_1 g_2)}}(h) = (g_1 g_2) \cdot h$$

$$= g_1 (g_2 h) = \underline{\underline{\rho_\ell(g_1) \circ \rho_\ell(g_2) h}}$$

ρ_r right multiplication

$$(\rho_r(g))(h) = h \cdot g$$

$$\rho_r(g_1 g_2)(h) = h(g_1 \cdot g_2)$$

$$= (hg_1) \cdot g_2$$

$$= \rho_r(g_2) \circ \rho_r(g_1)(h)$$

not an action!

Right way to do this:

$$\rho_r(g)(h) = h \cdot g^{-1}$$

$$\rho_r(g_1 g_2)(h) = h \cdot (g_1 g_2)^{-1}$$

$$= (h \cdot g_2^{-1}) \cdot (g_1)^{-1}$$

$$= \rho_r(g_1) \circ \rho_r(g_2)(h)$$

✓

Most important action:

Conjugation: $\rho_c : G \to \mathrm{Aut}(G)$

$$(\rho_c(g))(h) = g h g^{-1}$$

$$\rho_c = \rho_\ell \circ \rho_r$$
$$= \rho_r \circ \rho_\ell \qquad \rho_c(1)(h) = h$$

(1) $\rho_c(g_1 g_2)(h) = (g_1 g_2) h (g_1 g_2)^{-1}$

$$= g_1 (g_2 h g_2^{-1}) g_1^{-1} = \rho_c(g_1) \circ \rho_c(g_2)(h)$$

(2) $\rho_c(g)(h)$ is a gp. homomorphism!

$$\rho_c(g)(1) = g \cdot 1 \cdot g^{-1} = 1$$

$$\rho_c(g)(h_1 h_2) = g h_1 h_2 g^{-1}$$

$$= \underbrace{g h_1 g^{-1}}\underbrace{g h_2 g^{-1}}$$

$$= \rho_c(g)(h_1) \cdot \rho_c(g) h_2$$

Interested in orbits of

$h \in G$ under conjugation

Examples:

$C_d = \{1, x, x^2, \ldots, x^{d-1}\}$

is abelian, so

$(\rho_c(g))h = ghg^{-1} = gg^{-1}h$

$= h.$

(Conj classes of

C_d are $\{1\}, \{x\}, \{x^2\}, \ldots, \{x^{d-1}\}$)

$S_3 : \{1, (1\,2\,3), (1\,3\,2),$

$(1\,2), (1\,3), (2\,3)\}$

conjugacy class :

$\{1\}$, $\{(1\,2\,3), (1\,3\,2)\}$

$(g \cdot 1 \cdot g^{-1} = 1)$ $(1\,2)(1\,2\,3)(1\,2)$

$= (1\,3\,2)$

$\{(1\,2), (2\,3), (1\,3)\}$

$(1\,3)(1\,2)(1\,3) = (2\,3)$,

$(S_3 = D_6)$

$$D_{2n} = \left\{ \begin{matrix} 1, x, \ldots, x^{n-1} \\ \underline{y, xy, \ldots, x^{n-1}y} \end{matrix} \right\}$$

$$x^n = 1, \quad y^2 = 1, \quad \underline{\underline{yx = x^{-1}y}}$$

$$\{1\}, \{x, x^{-1}\}, \{x^2, x^{-2}\}, \ldots$$

$$yxy = (x^{-1}y)y = x^{-1}$$

$$\underline{\{y, yx^2, yx^4, \ldots\}}, \{yx, yx^3, \ldots\}$$

$$(x^{-1}y)x = (yx) \cdot x = yx^2$$

D_6 : $\{1\}, \{x, x^{-1}\}$

$\underline{\underline{}}$

$\{y, yx^2, yx^4 = yx\}$

S_3

$\{()\}, \{(1\,2\,3), (1\,3\,2)\}$

$\{(1\,2), (1\,3), (2\,3)\}$

D_8 : $\{1\}, \{x, x^3\}, \{x^2\}$

$\{y, yx^2\}, \{yx, yx^3\}$

Infinite dihedral gp!

$$O(2,\mathbb{R}) = \left\{ \begin{array}{c} \text{rotations} \\ + \\ \text{reflections} \end{array} \right\}$$

$$= SO(2,\mathbb{R}) \sqcup O(2,\mathbb{R})^-$$

subgp:

conjugacy classes:

- $O(2,\mathbb{R})^-$ is a single conj. class

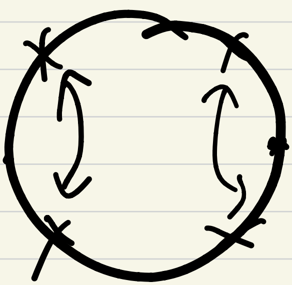
$$(\text{rot})\ \text{ref}_{\theta/2}\ (\text{rot}_{\theta'-\theta/2})^{-1} = \text{ref}_{\theta'/2}$$

· conj classes in $SO(2,\mathbb{R})$

$$= \{I_2\}, \{-I_2\}, \{rot_\theta, rot_{-\theta}\}$$

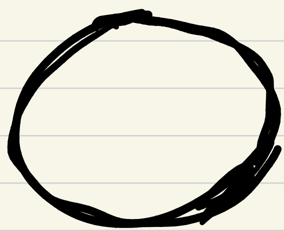
$$\{rot_\theta, rot_{-\theta}\} = \{rot_\theta, rot_\theta^{-1}\}$$

$SO(2,\mathbb{R})$



$\mathbb{R}$

$O^-(2,\mathbb{R})$



(S_n) Suppose we conjugate

τ by σ

- If $\tau(i) = j$, then

$(\sigma \tau \sigma^{-1})(\sigma(i))$

$= \sigma \tau(i) = \sigma(j)$.

Consequence:

$$\left[\sigma (1\ 2\ 3) \sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)) \right]$$