4800-12

I'll try to write a lot, Moses!

$$f : V \to V$$

linear map. Convert to a matrix by a choice of basis

$$\begin{tikzcd} V \arrow[r, "f"] & V \arrow[d, "g^{-1}"] \\ F^n \arrow[u, "g"] \arrow[r, "g^{-1} \circ f \circ g"] & F^n \end{tikzcd}$$

basis

= mult. by A

- f is **semi-simple** **if** g can be chosen so that A is **diagonal** (g is a basis of eigen-vectors)

Not every f is **semisimple**:

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} = A$$

is **not**.

$$c\ell(A) = \det \begin{bmatrix} (x-\lambda) & -1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ 0 & & & (x-\lambda) \end{bmatrix}$$

$$= (x-\lambda)^n$$

<u>So</u> the only <u>eigenvalue</u> of A is λ.

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\uparrow$

The only <u>eigenvector</u> is

$e_n = (0, \dots, 1)$ (and <u>multiples</u>).

$$A \cdot (v_1, \dots, v_n)$$

$$= (\lambda v_1, v_1 + \lambda v_2, \dots, v_{n-1} + \lambda v_n)$$

$$\| $$

$$\left(= (\lambda v_1, \lambda v_2, \dots, \lambda v_n) \right)$$

$$\Longrightarrow v_1 = v_2 = v_{n-1} = 0$$

$$\Longrightarrow \underline{(0, \dots, 0, v_n)}$$

$$\underbrace{}_{A}$$

$$(\lambda I_n - A)\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(\lambda I_n - A)\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\0\\ \vdots\\0\end{pmatrix}$$

$$A\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} = (\lambda, 1, 0, \dots, 0)$$

$$\underline{(\lambda I_n - A)}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} = (0, -1, 0, \dots, 0)$$

$$(\lambda I_n - A)\begin{pmatrix}0\\-1\\ \vdots\\0\end{pmatrix} = (0, 0, 1, 0, \dots)$$

$$(\lambda I_n - A)(e_1) = -e_2$$

$$(\lambda I_n - A)^2 e_1 = e_3$$

$$\vdots$$

$$(\lambda I_n - A)^{n-1} e_1 = (-1)^{n-1} e_n$$

$$(\lambda I_n - A)^n \cdot e_1 = 0$$

$$\underline{(\lambda I_n - A)^n} = O$$

(matrix)

$$\langle e_n \rangle = \ker(\underline{\lambda I_n} - A)$$

$$\langle e_{n-1}, e_n \rangle = \ker(\underline{\lambda I_n - A})^2$$

$$\vdots$$

$$\boxed{\langle e_1, \dots, e_n \rangle = \ker(\lambda I_n - A)^n}$$

In general (Jordan Normal Form)

Given $f : V \longrightarrow V$, $\exists$

basis $g : F^n \to V$ s.t.

$A = g^{-1} \circ f \circ g$ has the following form:

$$\begin{bmatrix} \begin{matrix}\lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda\end{matrix} & & \\ & \begin{matrix}\lambda_2 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_2\end{matrix} & \\ & & \ddots \end{bmatrix}$$

& all blocks

If there are no 1's above the diagonal, then f is semi-simple.

$$\begin{bmatrix} \boxed{\lambda} & & \\ & \boxed{\lambda} & \\ & & \boxed{\lambda} \end{bmatrix}$$

blocks of size one

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

In the Jordan normal form, require $F = \mathbb{C}$
(or another alg. closed field)

$$ch(A) = (x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2}$$

Ex:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Jordan

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

cx #'s

Semi-simple Examples

(1) If f has n distinct eigenvalues,

then
$$A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$f : \mathbb{C}^n \to \mathbb{C}^n$$

$$= \{ \mathbb{C}^{n^2} \text{ of } n \times n \text{ matrices} \}$$

(1) is dense in all matrices!

Fix A. Then $\forall \varepsilon$

$\exists A^\varepsilon$ s.t.

$$\max_{(i,j)} |a_{ij} - a^\varepsilon_{ij}| < \varepsilon$$

and A^ε has distinct

eigenvalues.

A

(2) Orthogonal matrices

(a) Extend scalars to $\mathbb{C}$.

$$ch(A) = \prod_{i=1}^{n} (x - \lambda_i)$$

complex #'s

$|\lambda_i| = 1$ (because

$$|\lambda \vec{v}| = |A(\vec{v})| = |\vec{v}|\ )$$

$\Rightarrow |\lambda| = 1$.

If $\lambda = \begin{cases} 1 = e^0 \\ -1 = e^{\pi i} \\ e^{i\theta} & 0, \pi \neq \theta \end{cases}$

$\lambda = e^{i\theta}$, then

$\bar{\lambda} = e^{-i\theta}$ is <u>also</u> an <u>eigenvalue</u>.

$A\vec{v} = \lambda\vec{v}$, then

$A \cdot \overline{\vec{v}} = \bar{\lambda}\,\overline{\vec{v}}$ $\quad \overline{\vec{v}}$ is

an eigenvector for $\bar{\lambda} = e^{-i\theta}$

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \qquad \vec{v}, \overline{\vec{v}}$$

Take instead

$$\frac{\vec{v} + \overline{\vec{v}}}{2}, \qquad \frac{\vec{v} - \overline{\vec{v}}}{2i}$$

$$\| \qquad\qquad \|$$

$$\vec{w}_1 \quad , \quad \vec{w}_2 \in \mathbb{R}^n$$

$$\left[\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \vec{w}_1, \vec{w}_2 \right]$$

(2) If $W \subseteq \mathbb{R}^n$ is a subspace fixed by A, then $W^\perp$ is also fixed.

$$A \rightsquigarrow \underline{(x-1)}^r\,\underline{(x+1)}^s \prod_{j=1}^{t} \underline{(x-e^{i\theta_j})}\,\underline{(x-e^{-i\theta_j})}$$

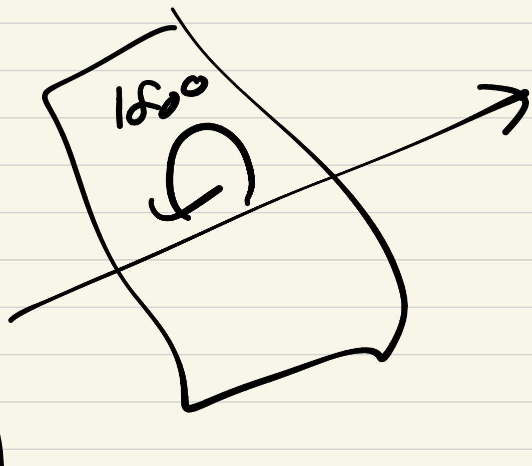
over $\mathbb{C}$:

$\mathbb{C}^1$:
$$\begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & \begin{bmatrix} e^{i\theta_1} & \\ & e^{-i\theta_1} \end{bmatrix} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \begin{matrix} e^{i\theta_t} & \\ & e^{-i\theta_t} \end{matrix} \end{bmatrix}$$

$\mathbb{R}^n$:
$$\begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & & & & & & \mathrm{rot}(\theta_1) & \\ & & & & & & & \ddots \\ & & & & & & & & \mathrm{rot}(\theta_t) \end{bmatrix} \qquad \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$\underline{\mathbb{R}^3}$, $\underline{\det = 1}$

$$\begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

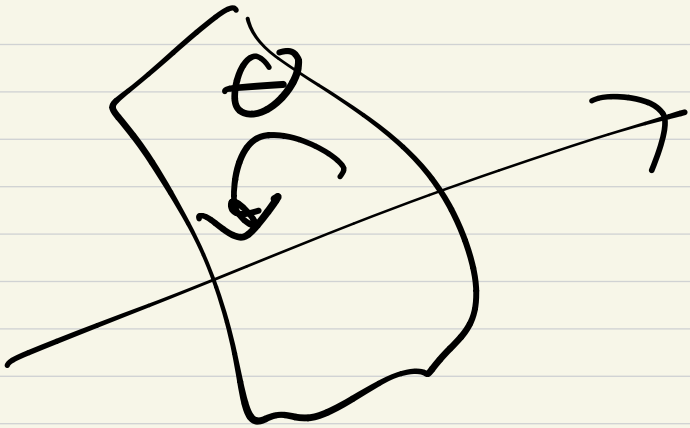


fixed

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \end{bmatrix}$$

$(\det = 1)$



<u>Unitary</u> ma<u>trices</u> (<u>analogous</u>)

Groups

Def: A group G is a set with an associative multiplication

$$\cdot \; G \times G \to G \,/$$

a 2-sided identity $1 \in G$

$$\forall g \quad 1 \cdot g = g = g \cdot 1$$

and 2-sided inverses.

Given g, $\exists!\, g^{-1} \in G$

s.t. $$g^{-1} \cdot g = 1 = g \cdot g^{-1}$$

Examples:

(1) $G = \{1, g, g^2, \ldots, g^{d-1}\}$

$(g^d = 1)$

is the cyclic gp. C_d.

(2) Dihedral grp

$$D_{2n} = \begin{Bmatrix} 1, x, \ldots, x^{n-1} \\ y, xy, \ldots, x^{n-1} \cdot y \end{Bmatrix}$$

$$\begin{pmatrix} x^n = 1, \; y^2 = 1, \\ xy = yx^{-1} \end{pmatrix}$$

(3)

(3) If $\mathcal{C}$ is a <u>category</u>,

then $\mathrm{Aut}(X)$ are a <u>gp</u>.

(symmetries of X as an object of $\underline{\mathcal{C}}$)

$$(\mathrm{Aut}(X), \circ, id_X)$$

$$\|$$

$$\{f : X \xrightarrow{\sim} X\}$$

(composition)

- $\circ$ is <u>associative</u>
- id_X is a 2-sided identity
- f has an inverse

- Permutations $\qquad \underset{\underline{\text{Sets}}}{} \operatorname{Aut}([n])$

- Euclidean groups:

$$\operatorname{Aut}(\mathbb{R}^n, d) \qquad \underline{\text{Met}}$$

Euclidean metric

- General linear gp:

$$GL(n, F) = \operatorname{Aut}(F^n)$$

$n \times n$ invertible matrices $\qquad \mathrm{Vec}_F$

$A_n \subset \mathrm{Aut}([n])$

(Alternating gp)

$A_4 \subset \mathrm{Aut}(\mathbb{R}^3, d)$

symmetries of Tetra

$A_5 \subset \mathrm{Aut}(\mathbb{R}^3, d)$

Dodec.

To define the category

$\underline{Gp}$ of groups,

need to define morphisms:

$$f : G \longrightarrow H$$ is

a group homomorphism if:

$$f(1) = 1$$

in G in H

$$f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$$

$$f(g_1^{-1}) = (f(g_1))^{-1}.$$

If $f: G \to H$ is a gp. hom. and a bijection,

then $f^{-1}: H \to G$

is a gp. homomorphism.

Examples:

$|\mathrm{Aut}([n])|$

$\parallel$

- $\mathrm{sgn}: \boxed{S_n} \longrightarrow \{\pm 1\}$

$\sigma \in S_n$

$\mathrm{sgn}(\sigma \circ \tau)$

$= \underline{\mathrm{sgn}(\sigma)} \cdot \underline{\mathrm{sgn}(\tau)}$

$\cdot\ \det : GL(n, F) \to (F^*, \cdot, 1)$

$\det(A \cdot B) = \det(A) \cdot \det(B)$

$\det(I_n) = 1$

Def: An *action* of G on X (in $\mathcal{C}$)

is a gp. hom.

$\cdot\ \rho : G \longrightarrow \mathrm{Aut}(X)$.

Examples:

- $\rho : A_4 \longrightarrow \mathrm{Aut}(\mathbb{R}^3, d)$

  A_4 ↑ rotations of Tet

- $\rho : C_n \longrightarrow \mathrm{Aut}(\mathbb{C})$

  $GL(1, \mathbb{C})$

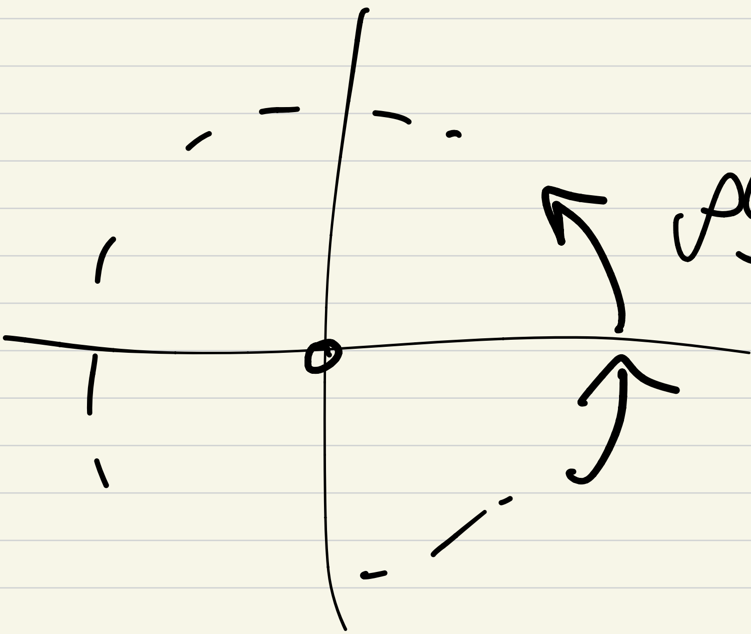
  $C_n = \{1, g, \dots, g^{d}\}$ $\qquad$ $\mathrm{Aut}(\mathbb{C}) = (\mathbb{C}^*, \cdot, 1)$

$$\rho(g) = e^{2\pi i/d}$$

$$\rho(g^2) = e^{4\pi i/d} \quad \dots$$

$$C_d = \{ 1, g, g^2, \dots, g^{d-1} \}$$

k



$$\rho(g) = e^{2\pi i/d}$$

$$\rho : C_d \longrightarrow \underline{\mathrm{Aut}(\mathbb{R}^2, d)}$$

$$\rho(g) = \begin{pmatrix} \cos(2\pi/d) & \cdot \\ \sin(2\pi/d) & \vdots \end{pmatrix}$$

$$ad : G \longrightarrow \underline{\mathrm{Aut}(G)}$$

in $\underline{\mathrm{Gp}}$