4800 - 10

Determinants etc.

Let $V = F^n$ (standard vector space)

$$V^* = \hom_F(V, F)$$

(Linear maps from V to F)

Basis: $x_1, \ldots, x_n$ for V^*

$$x_i(a_1, \ldots, a_n) = a_i$$

If: $f : V \to F$

then $\underline{f} = \underline{f(e_1)x_1 + \dots + f(e_n)x_n}$

$$f(a_1, \dots, a_n) = a_1 f(e_1) + \dots + a_n f(e_n)$$

$e_1, \dots, e_n \quad f_1, \dots, f_n$

If $f : V \times V \to F$

is <u>multilinear</u> <u>(2-tensor),</u>

then

$$f(e_i, e_j) = a_{ij}$$

are elements of an $n \times m$ <u>matrix</u>

(tensor)

Let $x_i \otimes x_j$ is

the 2-tensor that sets

$$x_i \otimes x_j\,(e_i, f_j) = 1$$

$$x_i \otimes x_j\,(e_k, f_\ell) = 0 \text{ for all others}$$

if $(k,\ell) \neq (i,j)$

A <u>2-tensor</u> is

a multi-linear map

$$f : V \times V \longrightarrow F$$

by which I mean:

$$f(\vec{v}_1 + \vec{v}_2, \vec{w}_1) = f(\vec{v}_1, \vec{w}_1) + f(\vec{v}_2, \vec{w}_1)$$

$$f(c\vec{v}, \vec{w}) = c f(\vec{v}, \vec{w})$$

$$f(\vec{v}, \vec{w}_1 + \vec{w}_2) = f(\vec{v}, \vec{w}_1) + f(\vec{v}, \vec{w}_2)$$

$$f(\vec{v}, c\vec{w}) = c f(\vec{v}, \vec{w}).$$

Basic 2-tensors:

$$(x_i \otimes x_j)((a_1, \dots, a_n), (b_1, \dots, b_n))$$

$$= a_i b_j.$$

$$f(e_i, e_j) = c_{ij}$$

$$\Leftrightarrow$$

$$f = \sum_{i=1}^{n} c_{ij} \, x_i \otimes x_j$$

Example: The dot product

$$\cdot : V \times V \longrightarrow F$$

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) =$$

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

$$x_1 \otimes x_1 + x_2 \otimes x_2 + \dots + x_n \otimes x_n$$

$$\boxed{\cdot = x_1 \otimes x_1 + \dots + x_n \otimes x_n}$$

$$\cdot = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

A 2-tensor is <u>symmetric</u> <u>if</u> the matrix (c_{ij}) is <u>symmetric</u> iff

f is a sum of multiples of 2-tensors of the form:

$$\underbrace{x_i \otimes x_i}_{A}, \quad \underline{x_i \otimes x_j + x_j \otimes x_i}$$

<u>alternating</u>:

$$x_i \otimes x_j - x_j \otimes x_i$$

Example of a symmetric 3-tensor

$$X_1 \otimes X_2 \otimes X_3 + X_2 \otimes X_1 \otimes X_3$$
$$+ X_3 \otimes X_2 \otimes X_1 + X_1 \otimes X_3 \otimes X_2$$
$$+ X_2 \otimes X_3 \otimes X_1 + X_3 \otimes X_1 \otimes X_2$$

alternating:

$$X_1 \otimes X_2 \otimes X_3 - X_2 \otimes X_1 \otimes X_3$$
$$\vdots$$

The <u>determinant</u> is <u>the</u> alternating n-tensor on F^n

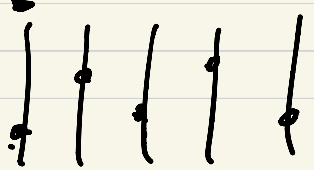
<u>with</u>

$$\det(\underbrace{e_1, e_2, \dots, e_n}_{I_n}) = 1$$

?!

<u>As a tensor, it is:</u>

$$\det(A) = \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$



multilinear!

Lemma:

(1) $\det(A^T) = \det(A)$

transpose

(2) det is an <u>alternating n-tensor</u>.

Pf:

$$\det(A^T) = \sum_{\sigma} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

$$= \sum_{\sigma^{-1}} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma^{-1}(i)}$$

$$= \sum_{\sigma} \underbrace{sgn(\sigma^{-1})}_{= sgn(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)} \quad \square$$

(2) $A_n \subset S_n$

(even permutations)

Choose $(i\ j) \in S_n$.

Then:

$$A_n \sqcup A_n \cdot (i\ j)$$

$$= S_n$$

$$|A_n| = \frac{1}{2} |S_n|$$

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) \cdot \prod a_{i,\sigma(i)}$$

$$= \sum_{\sigma \in A_n} \prod a_{i,\sigma(i)} - \sum_{\sigma \cdot (j\,k)} \prod a_{i,(\sigma \cdot (j\,k))(i)}$$

$(\operatorname{sgn} = +1)$ $(\operatorname{sgn} = -1)$

$$= \sum_{\sigma \in A_n} \prod a_{i,\sigma(i)} - \left(\sum_{\sigma \in A_n \setminus} \prod_{i \neq j,k} a_{i,\sigma(i)} \cdot a_{j\sigma(k)} \cdot a_{k\sigma(j)} \right)$$

$j \quad k$

Let $B = \begin{pmatrix} | & | \end{pmatrix}$. Then

the terms flip $\rightarrow$ (-1) sign. ☒

det is alternating &

$\det(e_1, \ldots, e_n) = \pm 1$.

Note: This tensor is the unique such tensor:

$$\sum_\sigma \operatorname{sgn}(\sigma)\, X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)}$$

All others are scalar multiples of this.

Thm: If A, B are two $n \times n$ matrices, then

$$\det(BA) = \det(B) \cdot \det(A)$$

Pf: Define an n-tensor by

$$T(A) = \det(B \cdot A)$$

fixed.

This is a tensor ✓

alternating ✓

$$\boxed{T(A) = \det(BA)}$$

It follows that

$$T(A) = b \cdot \underline{\det(A)}$$

$$T(I_n) = \det(B \cdot I_n)$$
$$= \det(B) \cdot 1$$

So $b = \det(B)$ $\square$

Change of basis:

$$f : V \to V$$

symmetry

$$\begin{tikzcd} V \arrow[r, "f"] & V \arrow[d, "g^{-1}"] \\ F^n \arrow[u, "g"] \arrow[r, "g^{-1} \circ f \circ g"] & F^n \end{tikzcd}$$

basis

$$f + \text{basis} \rightsquigarrow g^{-1} \circ f \circ g$$

$$= A \text{ matrix}$$

f + basis $\rightsquigarrow$ matrix

Challenge: Find the basis that fits $f: V \to V$ best

Example: If there is a basis of vectors such that

$$f(\vec{v}_i) = \lambda_i \vec{v}_i, \text{ then}$$

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

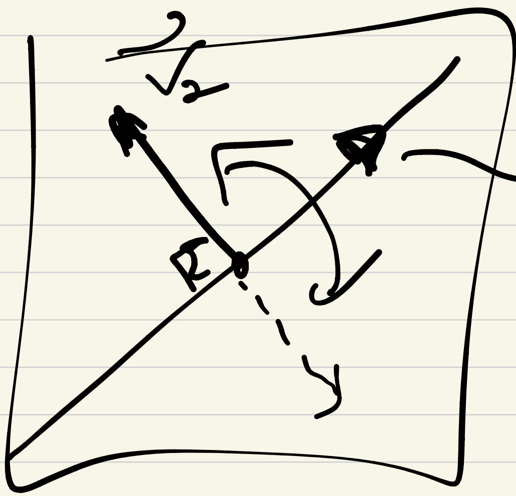
$$
\begin{tikzcd}
v_i & V \arrow[r, "f"] \arrow[d, phantom] & V \arrow[d] & \lambda_i v_i \arrow[d] \\
e_i \arrow[u] & F^n \arrow[u] \arrow[r, "A"] & F^n & \lambda_i e_i
\end{tikzcd}
$$

$$
\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}
$$

Such a transformation is called <u>semi-simple</u>.

Example:

Reflection of $\mathbb{R}^2$



$$f(\vec{v}_1) = \vec{v}_1$$

$$f(\vec{v}_2) = -\vec{v}_2$$

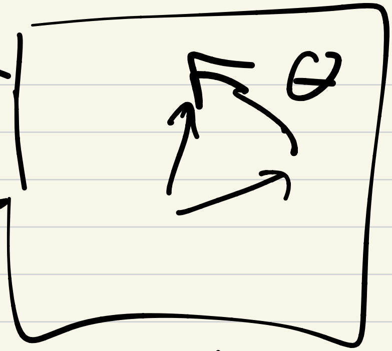
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\vec{v}_1$, $\vec{v}_2$

along the line of reflection

orth. to line of reflect

Rotation: $\mathbb{R}^2$



Looks like this has

<u>no</u> eigenvectors $(\neq \vec{0})$

Quest: <u>To find eigenvectors!</u>

$$\begin{tikzcd} V \arrow[r, "f"] & V \arrow[d, "S^{-1}"] \\ F^n \arrow[u, "S"] \arrow[r, "A"] & F^n \end{tikzcd}$$

Characteristic poly of A:

$$\det(\underset{\text{variable}}{x} \cdot I_n - A) = \underset{\text{polynomial in } x}{ch(A)}$$

Claim: The roots of $ch(A)$ are the eigenvalues λ_i of A.

$$\boxed{A\vec{v} = \underset{\text{eigenvalues}}{\lambda}\vec{v}}$$

$\underline{\lambda \text{ is}}$ a $\underline{\text{root}}$ of $\underline{ch(A)}$

$$\Updownarrow$$

$$\det(\lambda \cdot I_n - A) = 0$$

$$\Updownarrow$$

$$\ker(\underline{\lambda} \cdot I_n - A) \ni \vec{v}$$

$$\Updownarrow$$

$$\lambda \cdot \vec{v} - A\vec{v} = 0$$

$$\Updownarrow$$

$$A\vec{v} = \underline{\lambda} \vec{v}$$

Examples: 2x2 matrices

I_2: $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$

$$x \cdot I_2 - I_2 = \begin{bmatrix} x-1 & 0 \\ 0 & x-1 \end{bmatrix}$$

$$\underline{ch(I_2) = (x-1)^2}$$

Reflection: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$

across $\theta/2$

$$xI_2 - A = \begin{bmatrix} x-\cos\theta & -\sin\theta \\ -\sin\theta & x+\cos\theta \end{bmatrix}$$

$$ch(A) = x^2 - \cos^2\theta - \sin^2\theta = \underline{x^2-1}.$$

$$x^2-1=(x-1)(x+1)$$

$$\lambda=+1,\ -1$$

<u>To</u> <u>find</u> <u>eigenvectors</u>,

$$\ker(\underline{\lambda\cdot I_n-A})\neq 0$$

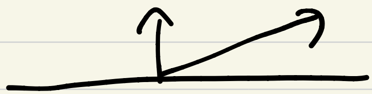
Find $\vec{v}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$ch(A) = \det \begin{bmatrix} x-1 & -1 \\ 0 & x-1 \end{bmatrix}$$

$$= \underline{\underline{(x-1)^2}}$$

eigenvalue 1:



only one eigenvector: $\Rightarrow b=0$

$$A \cdot e_1 = e_1 \qquad ae_1 + be_2 \;\; \|$$

$$A \cdot (ae_1 + be_2) = (a+b)e_1 + be_2$$