Second Midterm Due Wednesday, March 26 at 4:35.
Math 2270

1. Provide a complete and detailed proof of the following statement:

The vector space generated by the columns of an $m \times n$ matrix A has the same dimension as the vector space generated by the rows of A.

2. Let $\langle \underline{v}, \underline{w} \rangle$ be a positive definite inner product on a finite dimensional vector space V, and let:

$$||\underline{v}||^2 = \langle \underline{v}, \underline{v} \rangle$$

be the (square of the) corresponding norm.

(a) Prove that the norm satisfies the parallelogram law:

$$||\underline{v} + \underline{w}||^2 + ||\underline{v} - \underline{w}||^2 = 2(||\underline{v}||^2 + \underline{w}||^2)$$

for any pair of vectors $\underline{v}, \underline{w}$ in V.

(b) Prove that the scalar product satisfies:

$$\langle \underline{v}, \underline{w} \rangle = \frac{1}{4}(||\underline{v} + \underline{w}||^2 - ||\underline{v} - \underline{w}||^2)$$

Thus the norm *determines* the scalar product.

(c) What (if anything) goes wrong if $\langle \underline{v}, \underline{w} \rangle$ isn't positive definite?

3. Find an orthonormal basis for $\mathbb{R}^3$ with respect to the scalar product:

$$\langle \underline{v}, \underline{w} \rangle = \ {}^t\underline{v} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \underline{w}$$

Is this scalar product positive definite? Justify your answer.

4. Invert the matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

in two different ways. Show your work!

(a) Using row operations.

(b) Using determinants and minors (formula on page 177 of Lang).

5. (a) Find the eigenvalues of the transformation $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ for

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

(b) Find an orthonormal basis of eigenvectors for this transformation.

(c) Use (b) to diagonalize the matrix with a unitary matrix U. I.e.

$$^{t}UAU$$

should be a diagonal matrix.

6. (a) Compute the matrix for the transformation $L : V \to V$ where:

(i) $L = \frac{d}{dt}$ is the derivative.

(ii) V is the vector space with basis $\{\sin(t), \cos(t), t\cos(t), t\sin(t)\}$.

(b) Find all the eigenvalues of the matrix from (a).

(c) Does V have a basis consisting of eigenvectors for L?
Justify your answer.