of code

Pseudo-random numbers: a line at a

mostly

time

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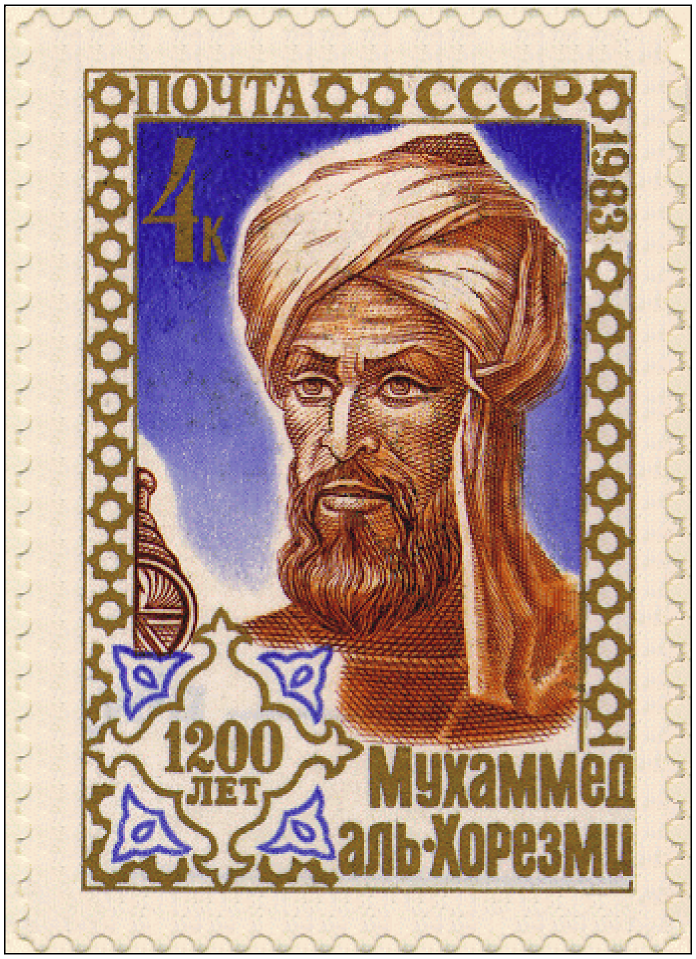
What are random numbers good for?

- Decision making (e.g., coin flip).
- Generation of numerical test data.
- Generation of unique cryptographic keys.
- Search and optimization via random walks.
- Selection: `quicksort` (C. A. R. Hoare, *ACM Algorithm 64: Quicksort*, Comm. ACM. **4**(7), 321, July 1961) was the first widely-used divide-and-conquer algorithm to reduce an $\mathcal{O}(N^2)$ problem to (on average) $\mathcal{O}(N \lg(N))$. Cf. Fast Fourier Transform (Gauss 1866 (Latin), Runge 1906, Danielson and Lanczos (crystallography) 1942, Cooley-Tukey 1965).

Historical note: al-Khwarizmi

Abu ’Abd Allah Muhammad ibn Musa al-Khwarizmi (ca. 780–850) is the father of *algorithm* and of *algebra*, from his book *Hisab Al-Jabr wal Mugabalah (Book of Calculations, Restoration and Reduction)*. He is celebrated in a 1200-year anniversary Soviet Union stamp:



What are random numbers good for? [cont.]

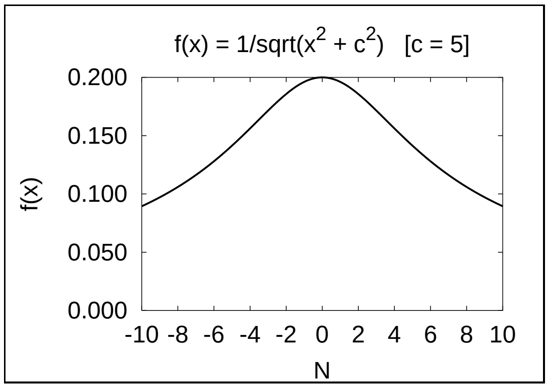
❑ Simulation.

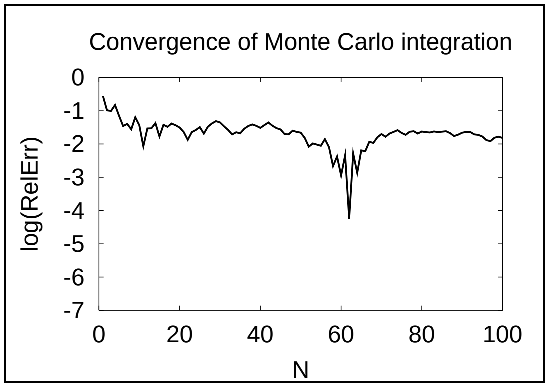
❑ Sampling: unbiased selection of random data in statistical computations (opinion polls, experimental measurements, voting, Monte Carlo integration, …). The latter is done like this (x_k is random in (a,b)):

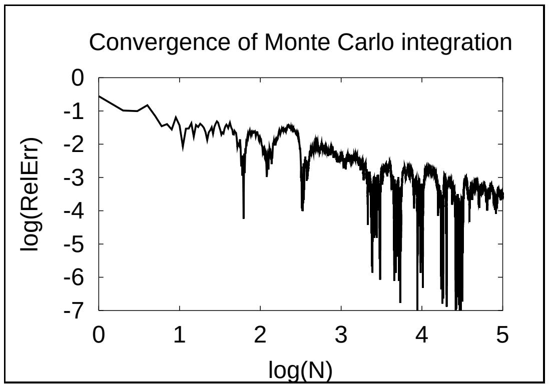
$$\int_a^b f(x)\,dx \approx \left(\frac{(b-a)}{N} \sum_{k=1}^{N} f(x_k) \right) + \mathcal{O}(1/\sqrt{N})$$

Monte Carlo integration

Here is an example of a simple, smooth, and exactly integrable function, and the relative error of its Monte Carlo integration.







When is a sequence of numbers random?

❏ Computer numbers are rational, with limited precision and range. Irrational and transcendental numbers are not represented.

❏ Truly random integers would have occasional repetitions, but most pseudo-random number generators produce a long sequence, called the *period*, of distinct integers: these cannot be random.

❏ It isn't enough to conform to an expected distribution: the *order* that values appear in must be haphazard.

❏ Mathematical characterization of randomness is possible, but difficult.

❏ The best that we can usually do is *compute statistical measures of closeness* to particular expected distributions.

Distributions of pseudo-random numbers

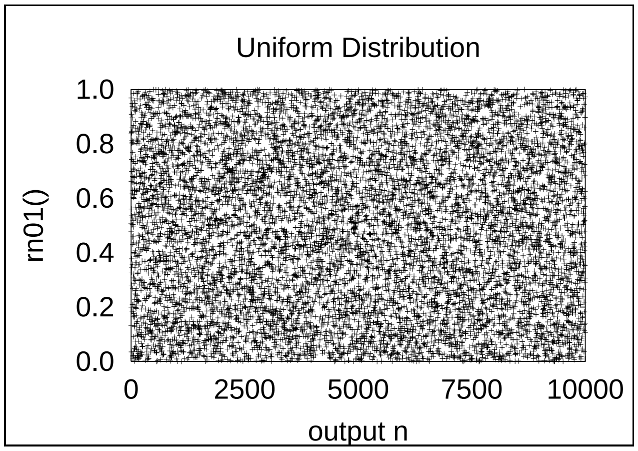
- ❏ Uniform (most common).
- ❏ Exponential.
- ❏ Normal (bell-shaped curve).
- ❏ Logarithmic: if $\mathrm{ran}()$ is uniformly-distributed in (a, b), define $\mathrm{randl}(x) = \exp(x\,\mathrm{ran}())$. Then $a\,\mathrm{randl}(\ln(b/a))$ is logarithmically distributed in (a, b). [Used for sampling in floating-point number intervals.]

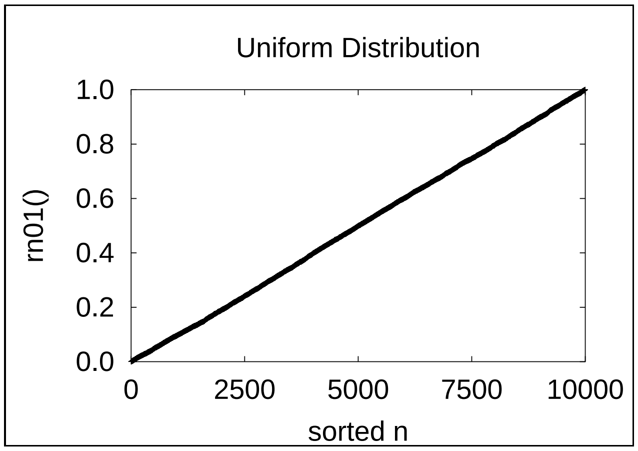
Distributions of pseudo-random numbers [cont.]

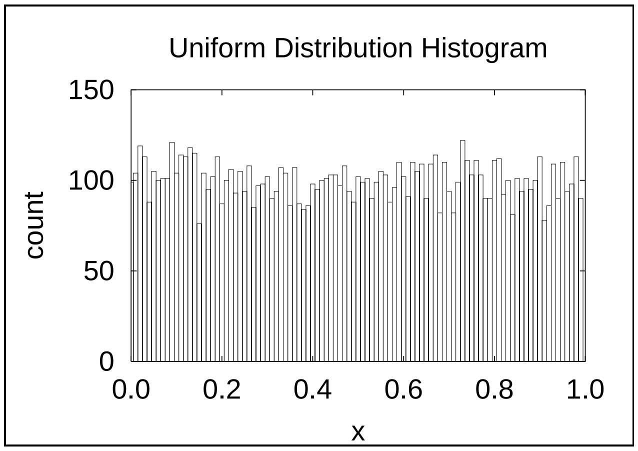
Sample logarithmic distribution:

```
% hoc
a = 1
b = 1000000
for (k = 1; k <= 10; ++k) \
    printf "%16.8f\n", a*randl(ln(b/a))
    664.28612484
 199327.86997895
 562773.43156449
  91652.89169494
     34.18748767
    472.74816777
     12.34092778
      2.03900107
  44426.83813202
     28.79498121
```

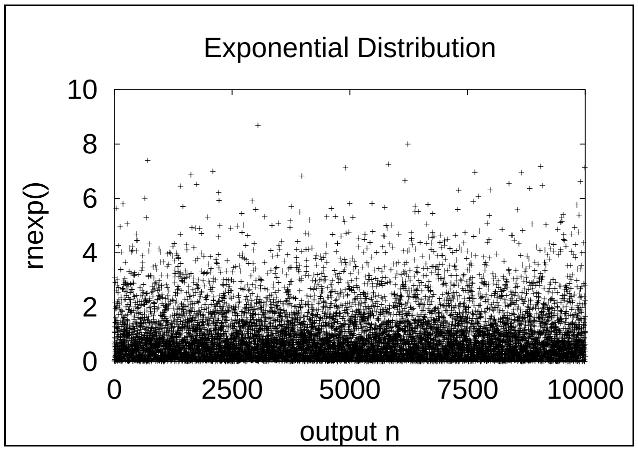
Uniform distribution

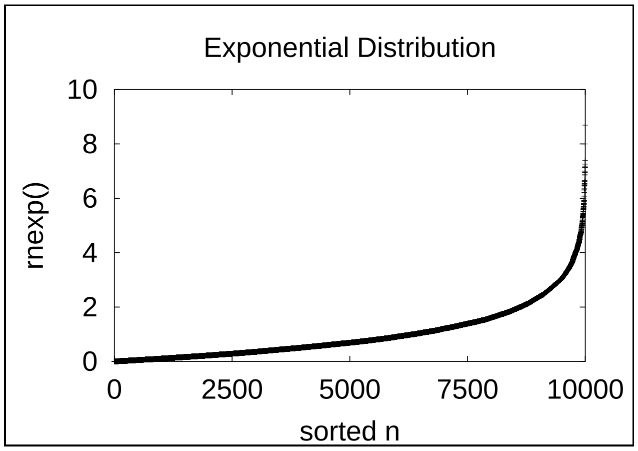


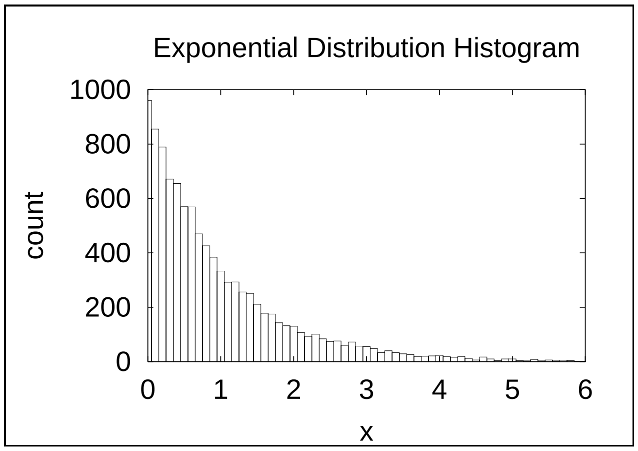




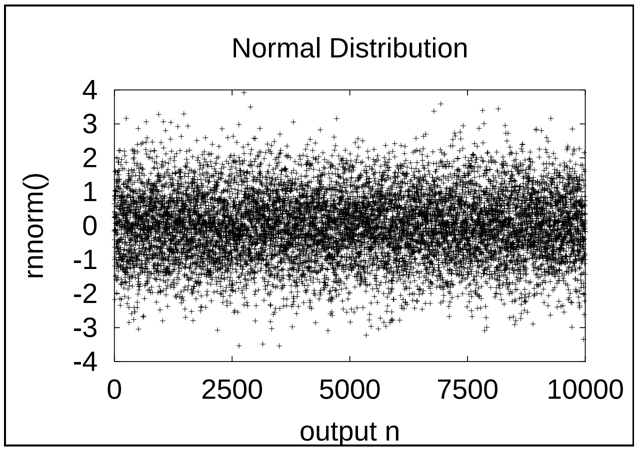
Exponential distribution

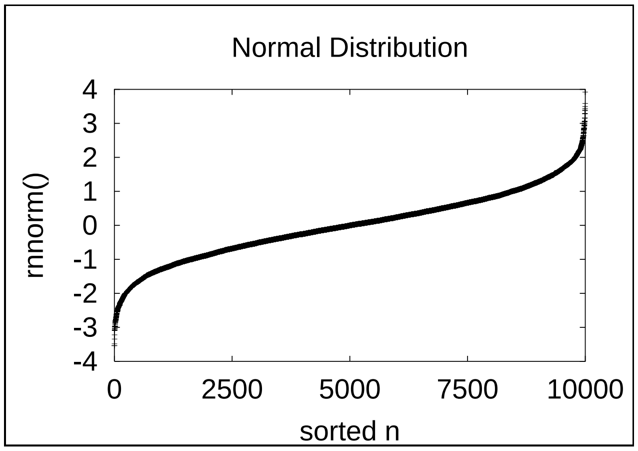


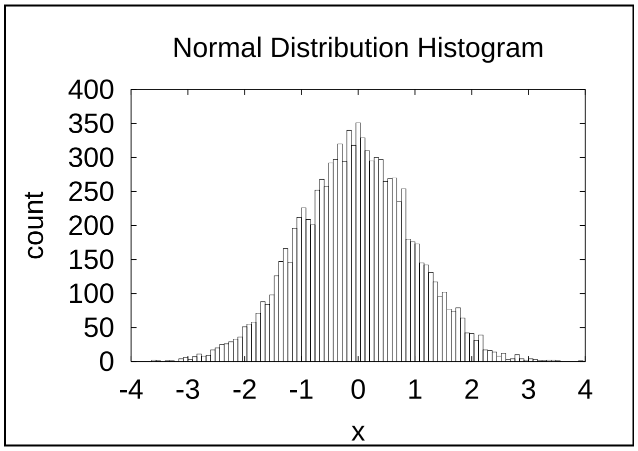




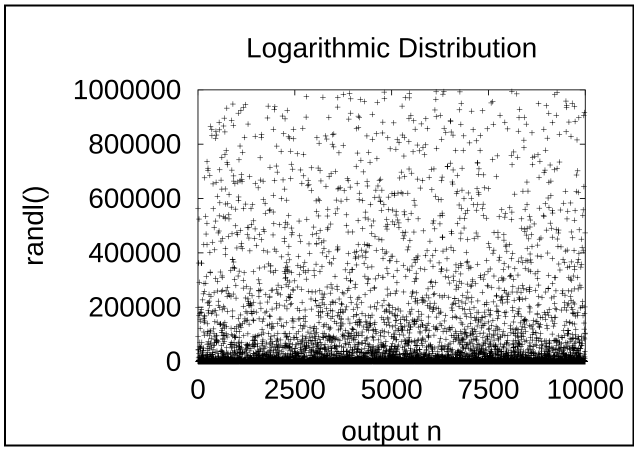
Normal distribution

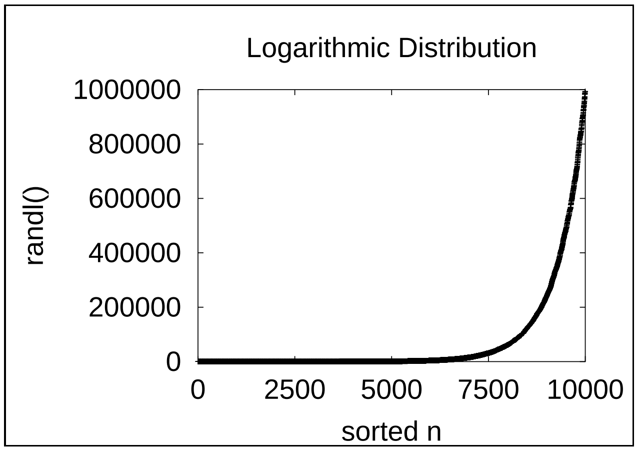


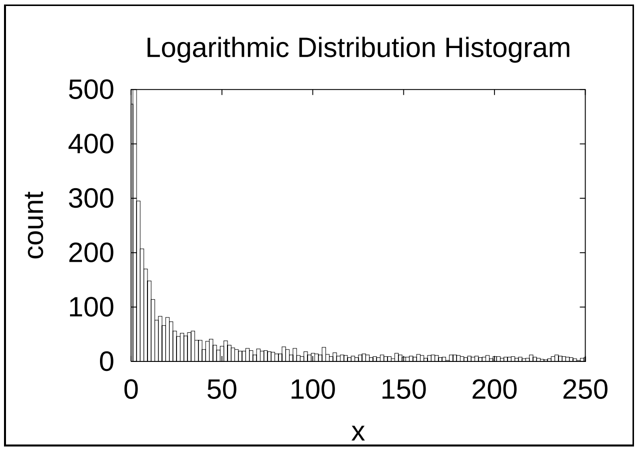




Logarithmic distribution







Goodness of fit: the χ^2 measure

Given a set of n *independent observations* with measured values M_k and expected values E_k, then $\sum_{k=1}^{n} |(E_k - M_k)|$ is a measure of goodness of fit. So is $\sum_{k=1}^{n} (E_k - M_k)^2$. Statisticians use instead a measure introduced by Pearson (1900):

$$\chi^2 \text{ measure} = \sum_{k=1}^{n} \frac{(E_k - M_k)^2}{E_k}$$

Equivalently, if we have s categories expected to occur with probability p_k, and if we take n samples, counting the number Y_k in category k, then

$$\chi^2 \text{ measure} = \sum_{k=1}^{s} \frac{(np_k - Y_k)^2}{np_k}$$

The theoretical χ^2 distribution depends on the number of degrees of freedom, and table entries look like this (boxes entries are referred to later):

Goodness of fit: the χ^2 measure [cont.]

D.o.f.	$p = 1\%$	$p = 5\%$	$p = 25\%$	$p = 50\%$	$p = 75\%$	$p = 95\%$	$p = 99\%$
$\boxed{\nu = 1}$	0.00016	0.00393	0.1015	$\boxed{0.4549}$	1.323	3.841	$\boxed{6.635}$
$\nu = 5$	0.5543	1.1455	2.675	4.351	6.626	11.07	15.09
$\boxed{\nu = 10}$	2.558	3.940	6.737	9.342	12.55	18.31	$\boxed{23.21}$
$\nu = 50$	29.71	34.76	42.94	49.33	56.33	67.50	76.15

This says that, e.g., **for $\nu = 10$, the probability that the χ^2 measure is no larger than 23.21 is 99%.**

For example, coin toss has $\nu = 1$: if it is not heads, then it must be tails.

```
for (k = 1; k <= 10; ++k) print randint(0,1), ""
0 1 1 1 0 0 0 0 1 0
```

This gave four 1s and six 0s:

$$\begin{aligned}\chi^2 \text{ measure} &= \frac{(10 \times 0.5 - 4)^2 + (10 \times 0.5 - 6)^2}{10 \times 0.5} \\ &= 2/5 \\ &= 0.40\end{aligned}$$

Goodness of fit: the χ^2 measure [cont.]

From the table, we expect a χ^2 measure no larger than 0.4549 half of the time, so our result is reasonable.

On the other hand, if we got nine 1s and one 0, then we have

$$\begin{aligned}\chi^2 \text{ measure} &= \frac{(10\times 0.5-9)^2+(10\times 0.5-1)^2}{10\times 0.5}\\ &= 32/5\\ &= 6.4\end{aligned}$$

This is close to the tabulated value 6.635 at $p = 99\%$. That is, we should only expect nine-of-a-kind about once in every 100 experiments.

If we had all 1s or all 0s, the χ^2 measure is 10 (probability $p = 0.998$).

If we had equal numbers of 1s and 0s, then the χ^2 measure is 0, indicating an exact fit.

Goodness of fit: the χ^2 measure [cont.]

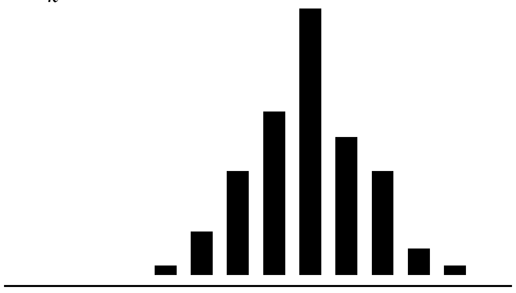
Let's try 100 similar experiments, counting the number of 1s in each experiment:

```
for (n = 1; n <= 100; ++n) {sum = 0
for (k = 1; k <= 10; ++k) sum += randint(0,1)
print sum, ""}
4 4 7 3 5 5 5 2 5 6 6 6 3 6 6 7 4 5 4 5 5 4
3 6 6 9 5 3 4 5 4 4 4 5 4 5 5 4 6 3 5 5 3 4
4 7 2 6 5 3 6 5 6 7 6 2 5 3 5 5 5 7 8 7 3 7
8 4 2 7 7 3 3 5 4 7 3 6 2 4 5 1 4 5 5 5 6 6
5 6 5 5 4 8 7 7 5 5 4 5
```

The measured frequencies of the sums are:

100 experiments

k	0	1	2	3	4	5	6	7	8	9	10
Y_k	0	1	5	12	19	31	16	12	3	1	0



Goodness of fit: the χ^2 measure [cont.]

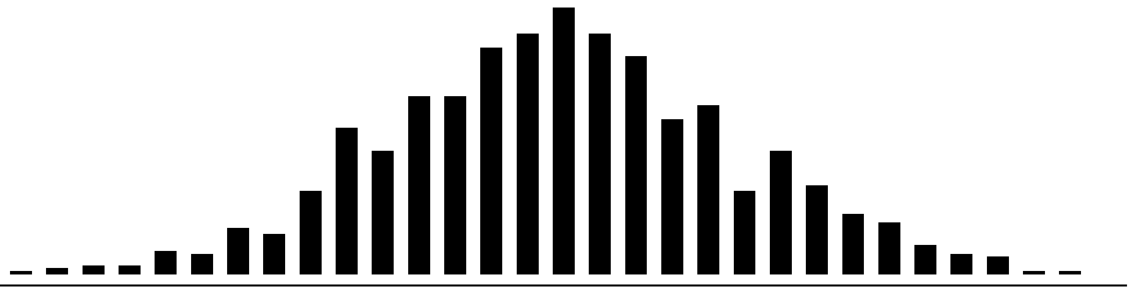
Notice that nine-of-a-kind occurred once each for 0s and 1s, as predicted.

A simple one-character change on the outer loop limit produces the next experiment:

1000 experiments

k	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
Y_k	1	2	3	3	8	7	16	14	29	51	43	62	62	79	84	93

k	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65
Y_k	84	76	54	59	29	43	31	21	18	10	7	6	1	1	0



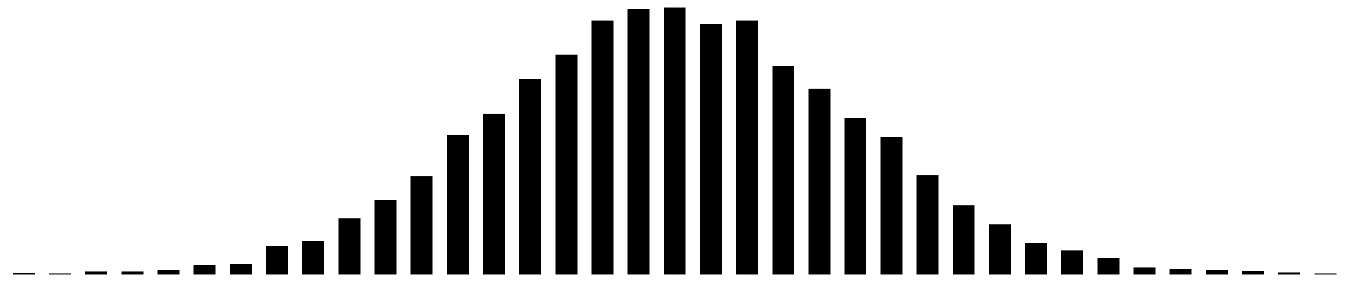
Goodness of fit: the χ^2 measure [cont.]

Another one-character change gives us this:

10 000 experiments

k	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
Y_k	0	0	3	1	7	7	12	27	30	85	99	168	224	295	420	484	588	663	766	799	804

k	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
Y_k	755	766	628	559	470	413	298	207	150	93	70	48	20	15	12	8	4	1	0	0

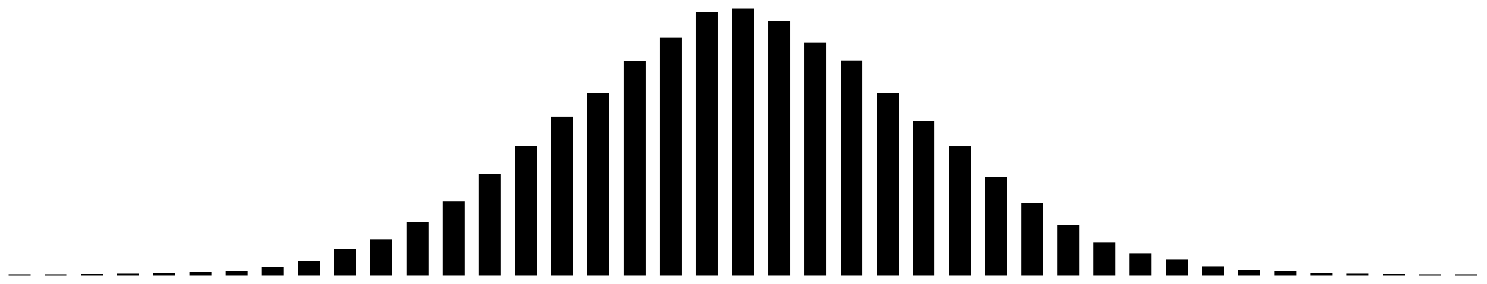


A final one-character change gives us this result for one million coin tosses:

100 000 experiments

k	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
Y_k	1	4	12	34	47	70	118	242	418	802	1086	1633	2259	3112	3987	4880	5609	6587	7320	8113	8227

k	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
Y_k	7828	7171	6607	5604	4740	3962	3029	2212	1544	996	654	474	257	141	107	43	37	21	5	5



Randomness of digits of π

Here are χ^2 results for the digits of π from recent computational records ($\chi^2(\nu = 9, P = 0.99) \approx 21.67$):

π

Digits	Base	χ^2	P(χ^2)
6B	10	9.00	0.56
50B	10	5.60	0.22
200B	10	8.09	0.47
1T	10	14.97	0.91
1T	16	7.94	0.46

$1/\pi$

Digits	Base	χ^2	P(χ^2)
6B	10	5.44	0.21
50B	10	7.04	0.37
200B	10	4.18	0.10

Whether the fractional digits of π, and most other transcendentals, are *normal* ($\approx$ equally likely to occur) is an outstanding unsolved problem in mathematics.

The Central-Limit Theorem

The famous **Central-Limit Theorem** (de Moivre 1718, Laplace 1810, and Cauchy 1853), says:

> A suitably normalized sum of independent random variables is likely to be normally distributed, as the number of variables grows beyond all bounds. It is not necessary that the variables all have the same distribution function or even that they be wholly independent.
>
> — I. S. Sokolnikoff and R. M. Redheffer
> *Mathematics of Physics and Modern Engineering*, 2nd ed.

The Central-Limit Theorem [cont.]

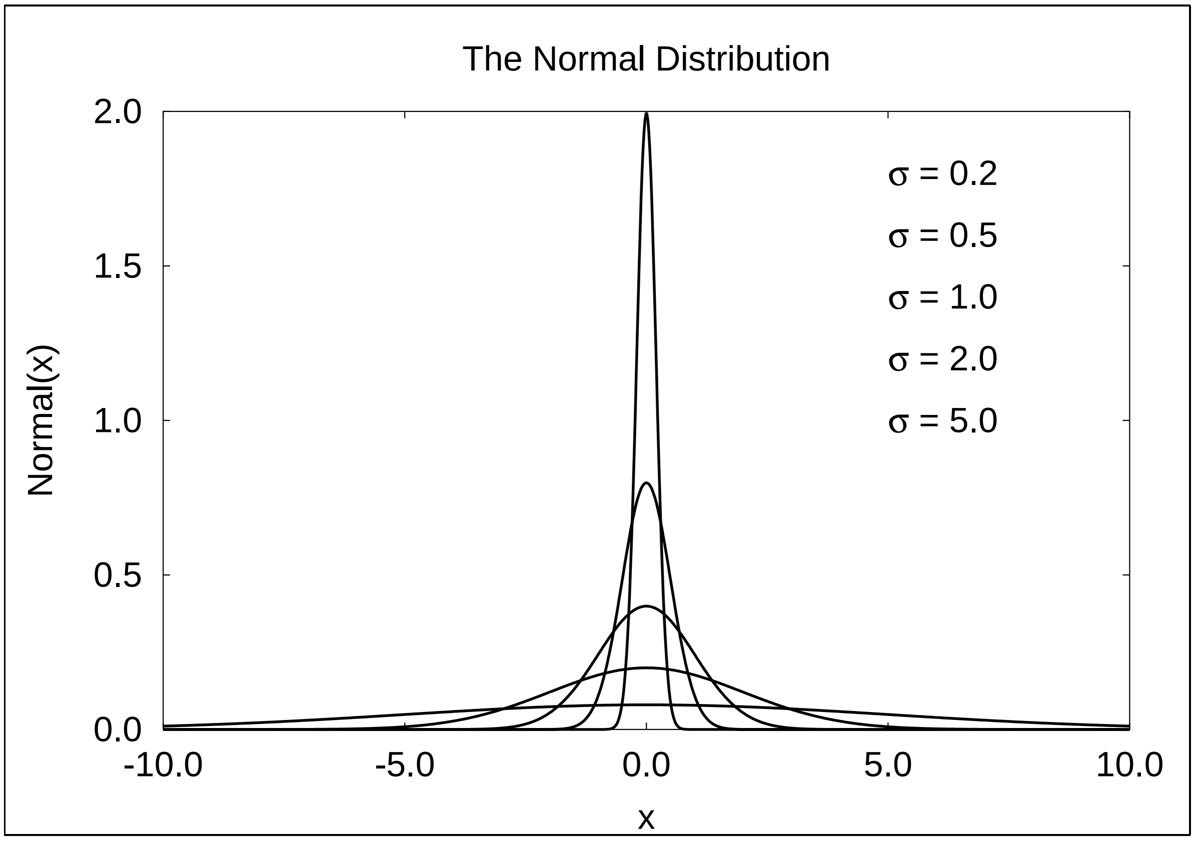
In mathematical terms, this is

$$P(n\mu + r_1\sqrt{n} \le X_1 + X_2 + \cdots + X_n \le n\mu + r_2\sqrt{n})$$
$$\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{r_1}^{r_2} \exp(-t^2/(2\sigma^2))dt$$

where the X_k are independent, identically distributed, and bounded random variables, μ is their *mean value*, σ is their *standard deviation*, and σ^2 is their *variance*.

The Central-Limit Theorem [cont.]

The integrand of this probability function looks like this:



The Central-Limit Theorem [cont.]

The normal curve falls off very rapidly. We can compute its area in $[-x, +x]$ with a simple midpoint quadrature rule like this:

```
func f(x) {global sigma;
           return (1/(sigma*sqrt(2*PI)))*
                  exp(-x*x/(2*sigma**2))}
func q(a,b){n = 10240; h = (b - a)/n; s = 0;
            for (k = 0; k < n; ++k)
                 s += h*f(a + (k + 0.5)*h);
            return s}
```

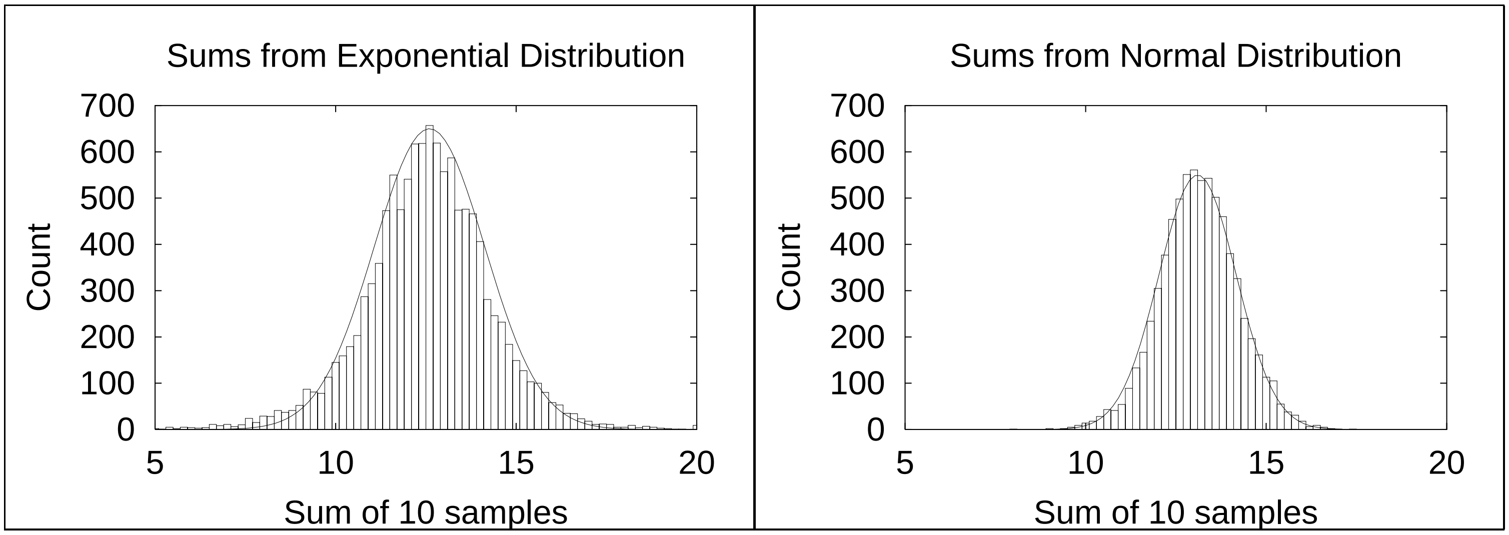
The Central-Limit Theorem [cont.]

```
sigma = 3
for (k = 1; k < 8; ++k)
    printf "%d  %.9f\n", k, q(-k*sigma,k*sigma)
1  0.682689493
2  0.954499737
3  0.997300204
4  0.999936658
5  0.999999427
6  0.999999998
7  1.000000000
```

In computers, 99.999% (five 9's) availability is 5 minutes downtime per year. In manufacturing, Motorola's 6σ reliability with 1.5σ drift is about 3.4 defects per million (from $q(4.5 * \sigma)/2$).

The Central-Limit Theorem [cont.]

It is remarkable that the Central-Limit Theorem applies also to nonuniform distributions: here is a demonstration with sums from exponential and normal distributions. Superimposed on the histograms are rough fits by eye of normal distribution curves $650\exp(-(x - 12.6)^2/4.7)$ and $550\exp(-(x - 13.1)^2/2.3)$.



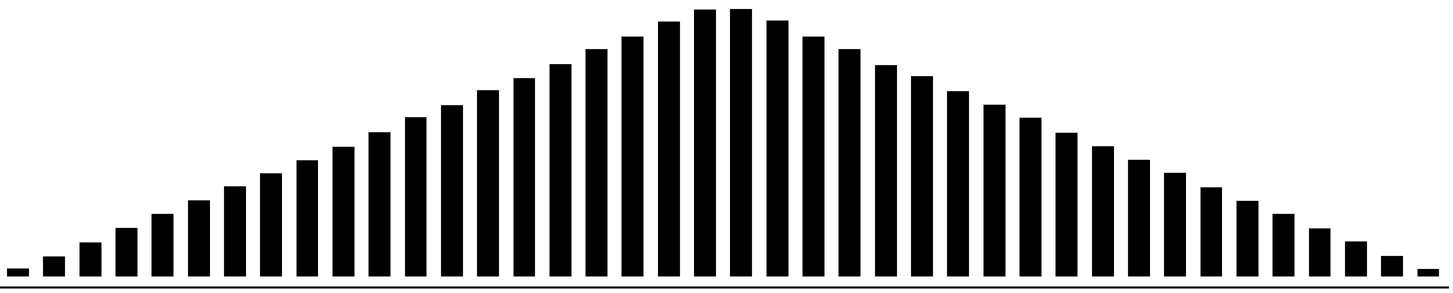
The Central-Limit Theorem [cont.]

Not everything looks like a normal distribution. Here is a similar experiment, using *differences* of successive pseudo-random numbers, bucketizing them into 40 bins from the range $[-1.0, +1.0]$:

10 000 experiments (counts scaled by 1/100)

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
					1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
	1	3	6	8	1	3	6	8	1	3	6	9	1	3	6	8	1	3	6	8
Y_k	3	5	1	8	3	8	3	7	1	6	2	0	2	9	1	7	4	7	4	7

k	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
	4	4	4	4	3	3	3	3	2	2	2	2	1	1	1	1				
	8	6	3	1	8	6	3	1	8	6	3	1	8	6	3	1	8	6	3	1
Y_k	7	7	7	4	5	5	7	2	8	1	6	2	8	2	7	3	7	3	6	2



This one is known from theory: it is a *triangular* distribution. A similar result is obtained if one takes pair sums instead of differences.

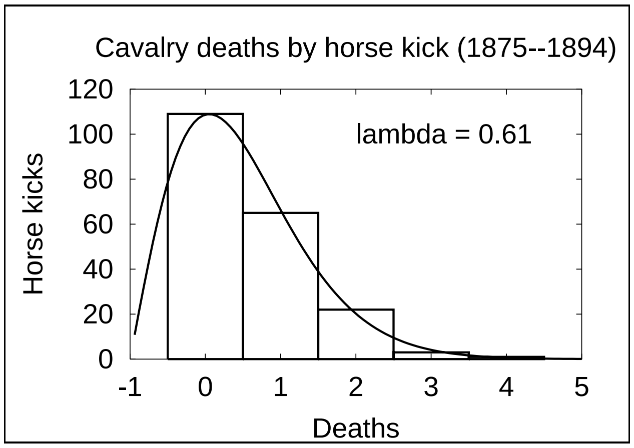
Digression: Poisson distribution

The *Poisson* distribution arises in time series when the probability of an event occurring in an arbitrary interval is proportional to the length of the interval, and independent of other events:

$$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

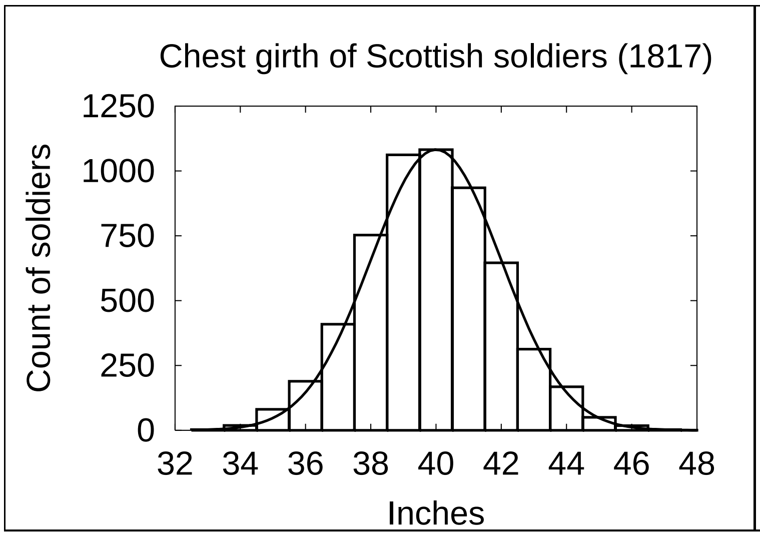
In 1898, Ladislaus von Bortkiewicz collected Prussian army data on the number of soldiers killed by horse kicks in 10 cavalry units over 20 years: 122 deaths, or an average of $122/200 = 0.61$ deaths per unit per year.

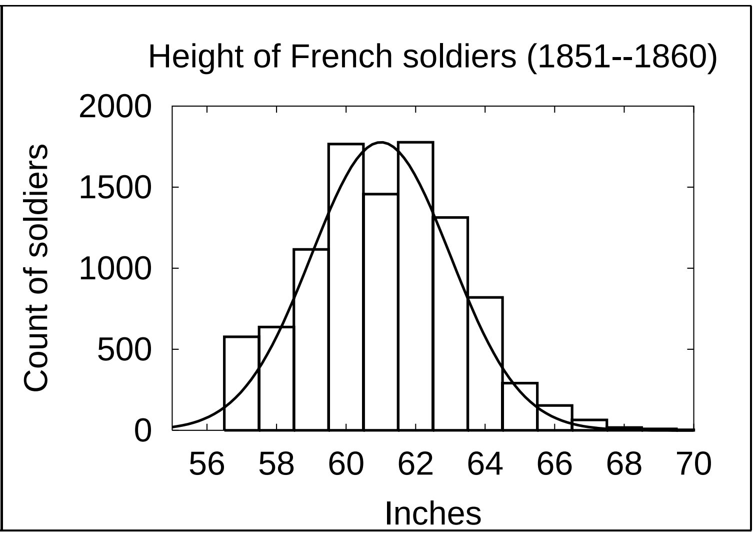
	$\lambda = 0.61$	
Deaths	Kicks (actual)	Kicks (Poisson)
0	109	108.7
1	65	66.3
2	22	20.2
3	3	4.1
4	1	0.6

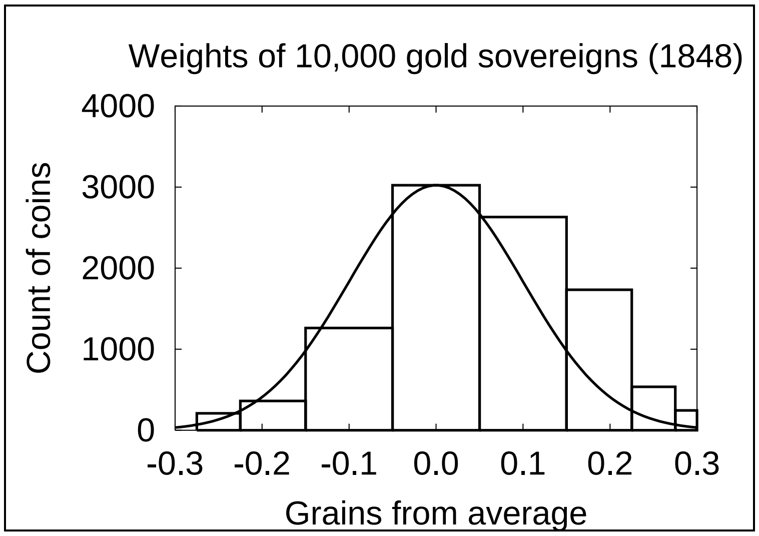


The Central-Limit Theorem [cont.]

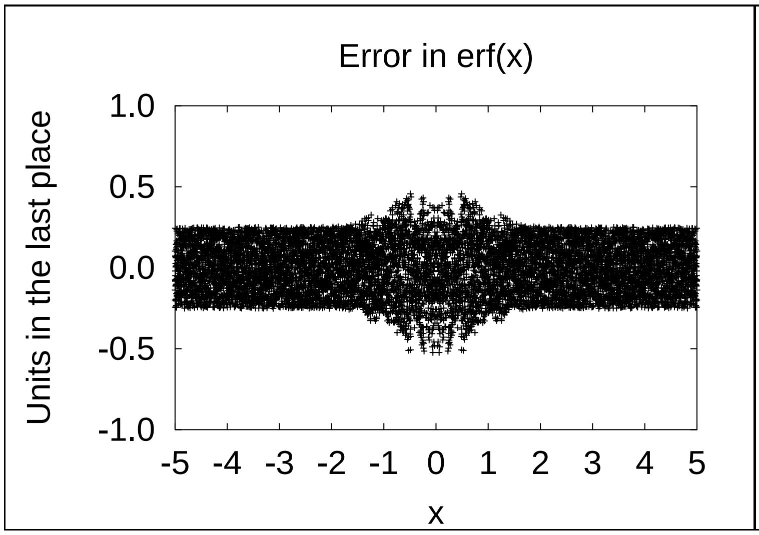
Measurements of physical phenomena often form normal distributions:

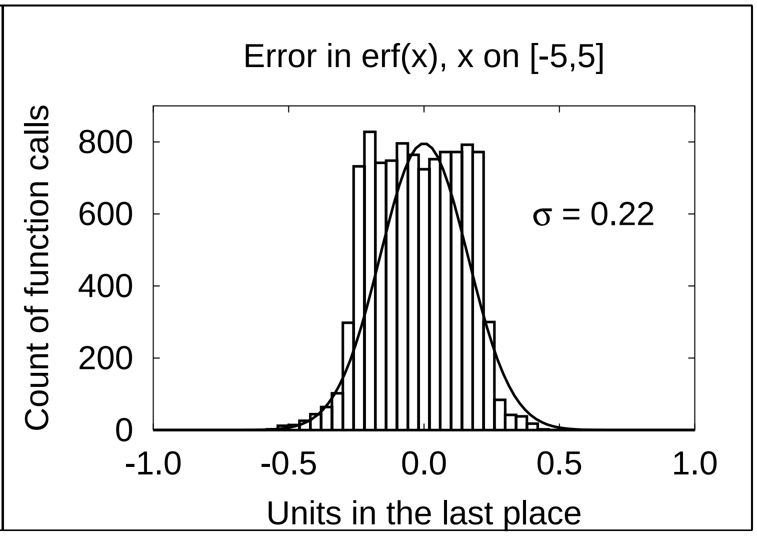


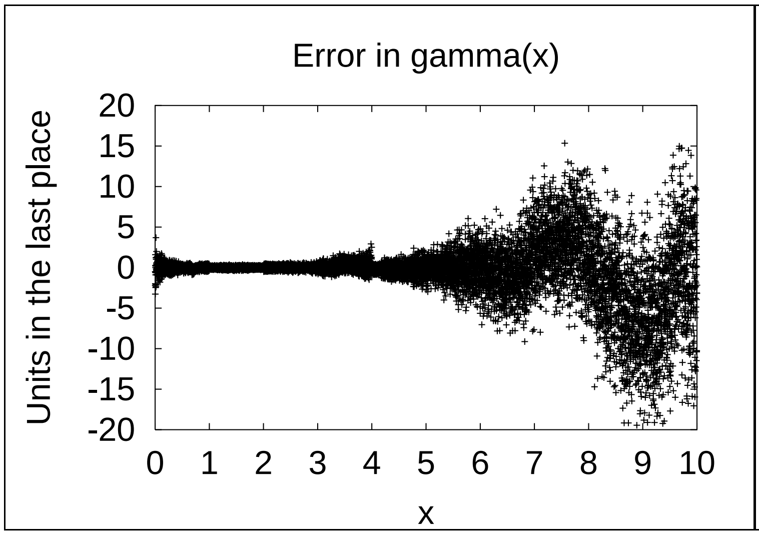


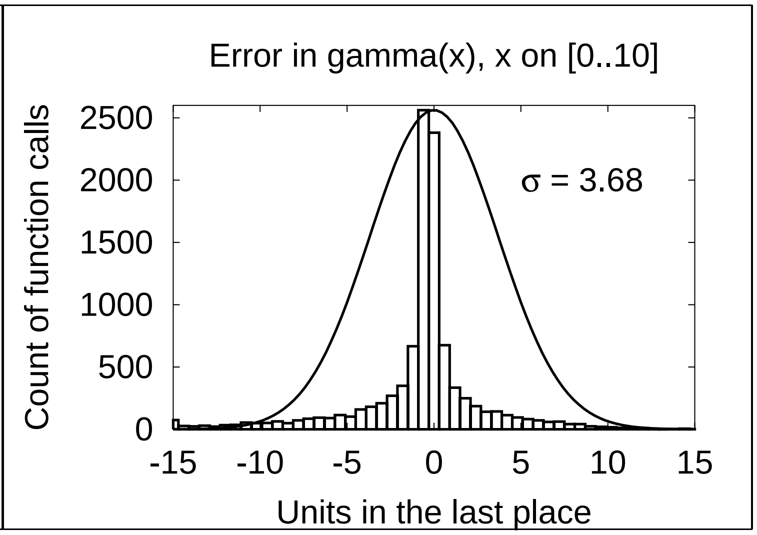


The Central-Limit Theorem [cont.]

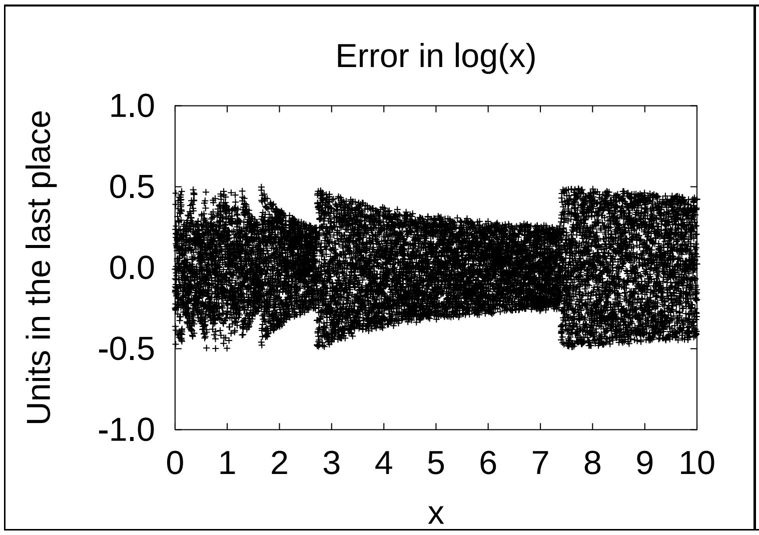


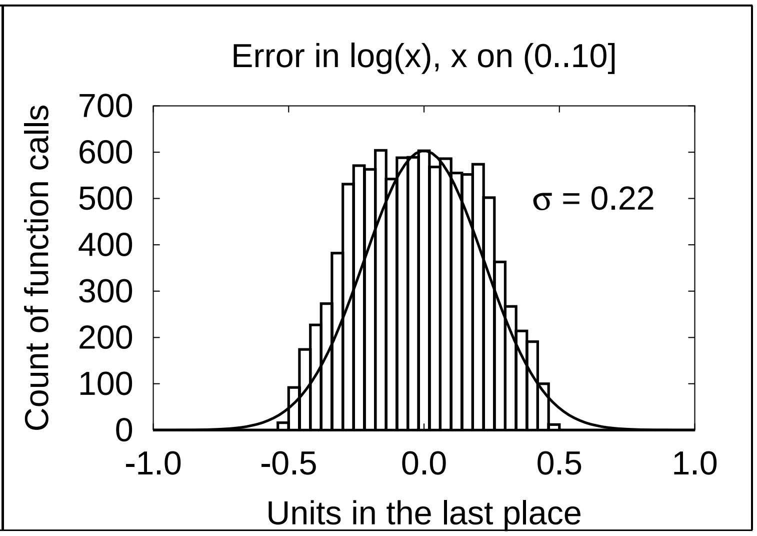


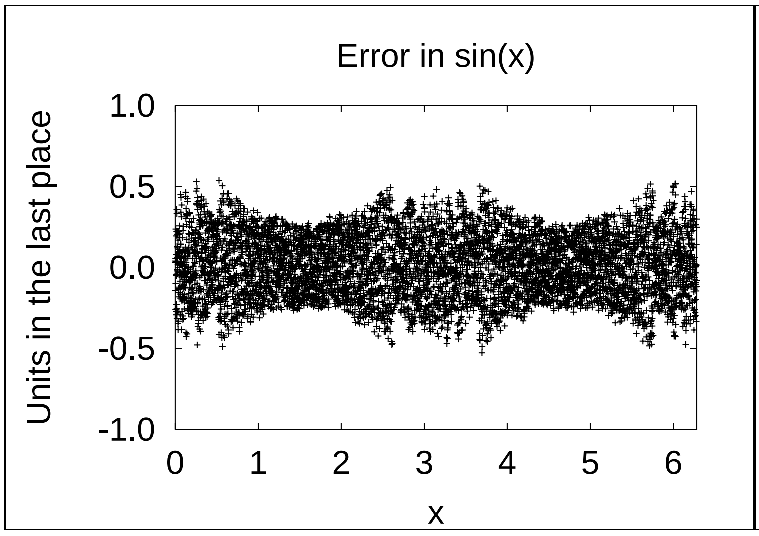


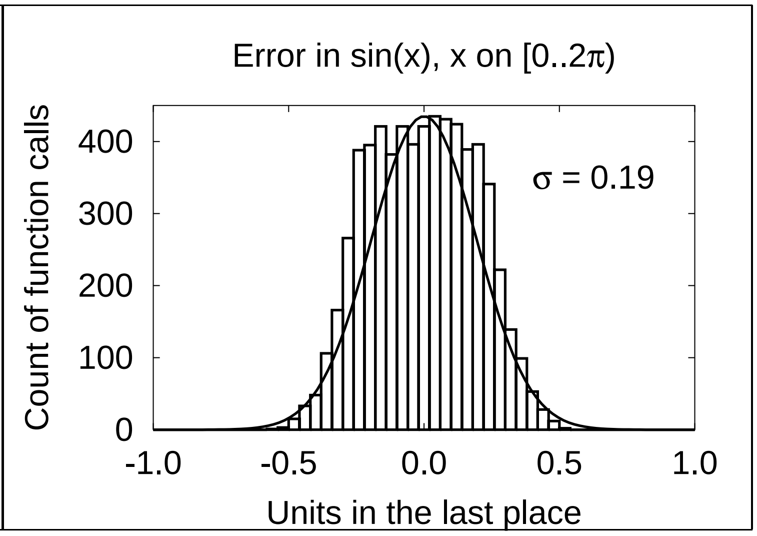


The Central-Limit Theorem [cont.]









Any one who considers arithmetical methods of producing random numbers is, of course, in a state of sin.
— John von Neumann (1951)

[from *The Art of Computer Programming, Vol. 2, Seminumerical Algorithms*, 3rd ed., p. 1]

He talks at random; sure, the man is mad.
— Margaret, daughter to Reignier,
afterwards married to King Henry
in William Shakespeare's *1 King Henry VI*, Act V,
Scene 3 (1591)

A random number generator chosen at random isn't very random.
— Donald E. Knuth
[*The Art of Computer Programming, Vol. 2, Seminumerical Algorithms*, 3rd ed., p. 384]

How do we generate pseudo-random numbers?

- ❑ Linear-congruential generators (most common): $r_{n+1} = (ar_n + c) \bmod m$, for integers a, c, and m, where $0 < m$, $0 \le a < m$, $0 \le c < m$, with starting value $0 \le r_0 < m$.

- ❑ Fibonacci sequence (bad!): $r_{n+1} = (r_n + r_{n-1}) \bmod m$.

- ❑ Additive (better): $r_{n+1} = (r_{n-\alpha} + r_{n-\beta}) \bmod m$.

- ❑ Multiplicative (bad): $r_{n+1} = (r_{n-\alpha} \times r_{n-\beta}) \bmod m$.

- ❑ Shift register: $r_{n+k} = \sum_{i=0}^{k-1}(a_i r_{n+i} \pmod 2) \qquad (a_i = 0, 1)$.

How do we generate pseudo-random numbers? [cont.]

Given an integer $r \in [A, B)$, $x = (r - A)/(B - A + 1)$ is on $[0, 1)$.

However, interval reduction by $A + (r - A) \bmod s$ to get a distribution in (A, C), where $s = (C - A + 1)$, is possible only for certain values of s. Consider reduction of $[0, 4095]$ to $[0, m]$, with $m \in [1, 9]$: we get equal distribution of remainders only for $m = 2^q - 1$:

	m	counts of remainders $k \bmod (m+1)$, $k \in [0, m]$									
OK	1	2048	2048								
	2	**1366**	1365	1365							
OK	3	1024	1024	1024	1024						
	4	**820**	819	819	819	819					
	5	**683**	**683**	**683**	**683**	682	682				
	6	**586**	585	585	585	585	585	585			
OK	7	512	512	512	512	512	512	512	512		
	8	**456**	455	455	455	455	455	455	455	455	
	9	**410**	**410**	**410**	**410**	**410**	**410**	409	409	409	409

How do we generate pseudo-random numbers? [cont.]

Samples from other distributions can usually be obtained by some suitable transformation. Here is the simplest generator for the normal distribution, assuming that randu() returns uniformly-distributed values on $(0, 1]$:

```
func randpmnd() \
{
  ## Polar method for random deviates
  ## Algorithm P, p. 122, from Donald E. Knuth,
  ## The Art of Computer Programming, 3/e, 1998

    while (1) \
    {
        v1 = 2*randu() - 1
        v2 = 2*randu() - 1
        s = v1*v1 + v2*v2
        if (s < 1) break
    }

    return (v1 * sqrt(-2*ln(s)/s))
}
```

Period of a sequence

All pseudo-random number generators eventually reproduce the starting sequence; the *period* is the number of values generated before this happens. Good generators are now known with periods $> 10^{449}$ (e.g., Matlab's `rand()`).

Reproducible sequences

In computational applications with pseudo-random numbers, it is *essential* to be able to reproduce a previous calculation. Thus, generators are required that can be set to a given *initial seed*:

```
% hoc
for (k = 0; k < 3; ++k) \
{
    setrand(12345)
    for (n = 0; n < 10; ++n) print int(rand()*100000),""
    println ""
}
88185 5927 13313 23165 64063 90785 24066 37277 55587 62319
88185 5927 13313 23165 64063 90785 24066 37277 55587 62319
88185 5927 13313 23165 64063 90785 24066 37277 55587 62319
```

Reproducible sequences [cont.]

```
for (k = 0; k < 3; ++k) \
{
    ## setrand(12345)
    for (n = 0; n < 10; ++n) print int(rand()*100000),""
    println ""
}
36751 37971 98416 59977 49189 85225 43973 93578 61366 54404
70725 83952 53720 77094 2835 5058 39102 73613 5408 190
83957 30833 75531 85236 26699 79005 65317 90466 43540 14295
```

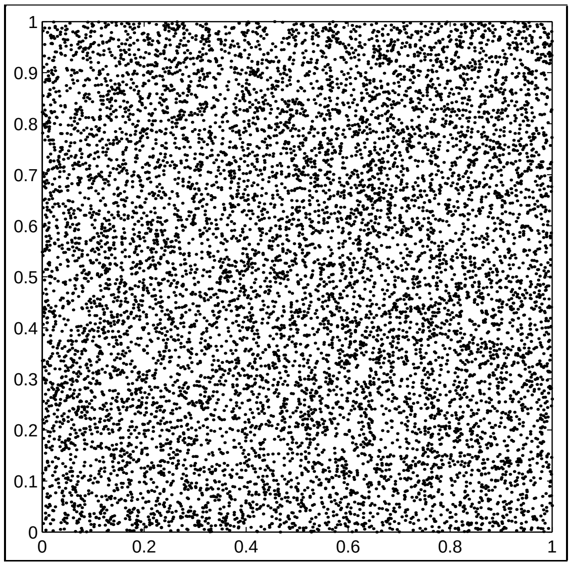
In practice, software must have its own source-code implementation of the generators: vendor-provided ones do not suffice.

The correlation problem

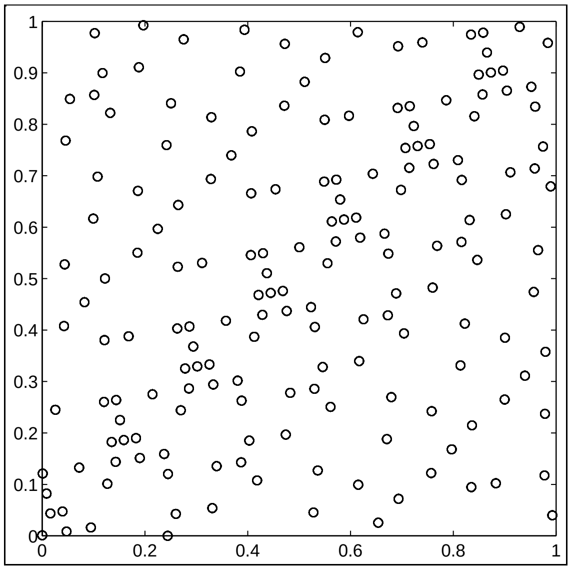
Random numbers fall mainly in the planes
— George Marsaglia (1968)

Linear-congruential generators are known to have correlation of successive numbers: if these are used as coordinates in a graph, one gets patterns, instead of uniform grey. The number of points plotted in each is the same in both graphs:

Good



Bad



The correlation problem [cont.]

The good generator is Matlab’s rand(). Here is the bad generator:

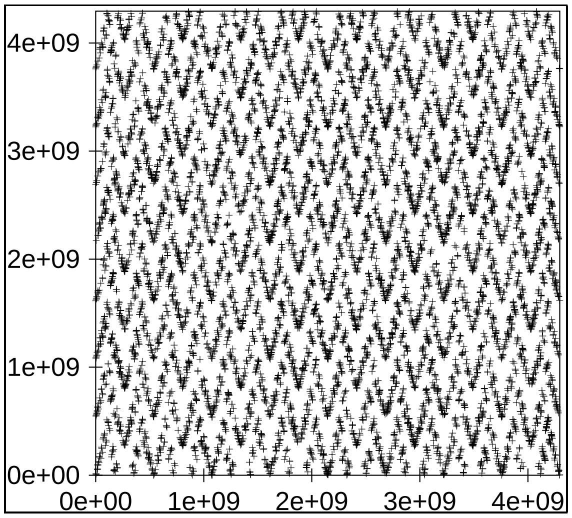
```
% hoc
func badran() { global A, C, M, r;
           r = int(A*r + C) % M; return r }
M = 2^15 - 1; A = 2^7 - 1 ; C = 2^5 - 1
r = 0 ; r0 = r ; s = -1 ; period = 0

while (s != r0) {period++; s = badran();
                 print s, "" }
    31 3968 12462 9889 10788 26660 ...
    22258 8835 7998 0

# Show the sequence period
println period
    175

# Show that the sequence repeats
for (k = 1; k <= 5; ++k) print badran(),""
    31 3968 12462 9889 10788
```

The correlation problem [cont.]

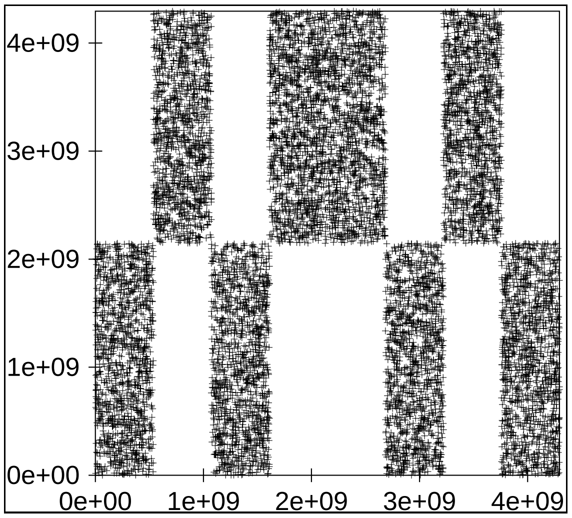
Marsaglia’s (2003) family of xor-shift generators:

```
y ^= y << a; y ^= y >> b; y ^= y << c;
```

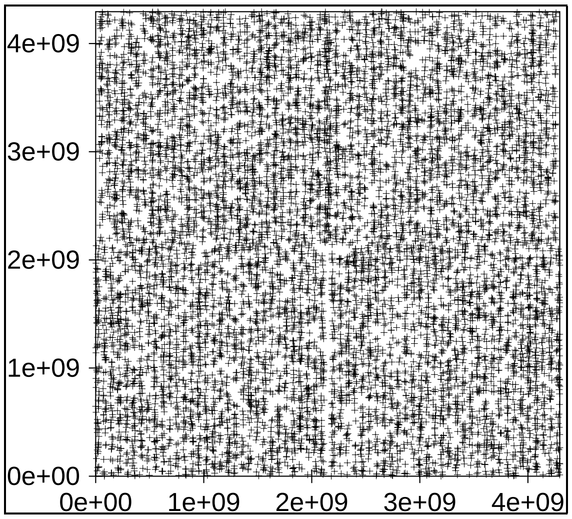
l.003



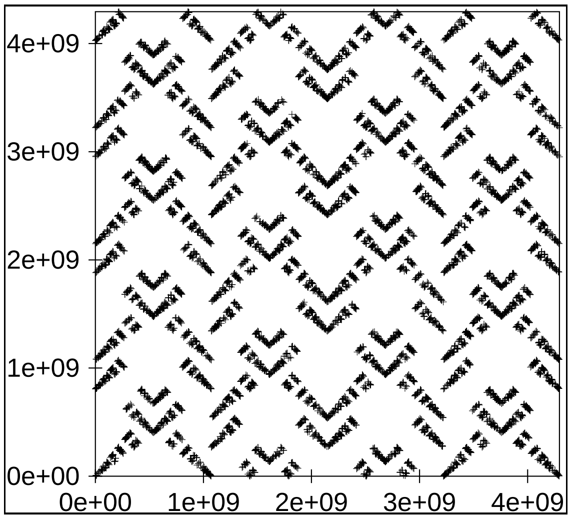
l.007



l.028



l.077



Generating random integers

When the endpoints of a floating-point uniform pseudo-random number generator are uncertain, generate random integers in [low,high] like this:

```
func irand(low, high) \
{
    # Ensure integer endpoints
    low = int(low)
    high = int(high)

    # Sanity check on argument order
    if (low >= high) return (low)

    # Find a value in the required range
    n = low - 1
    while ((n < low) || (high < n)) \
        n = low + int(rand() * (high + 1 - low))

    return (n)
}

for (k = 1; k <= 20; ++k) print irand(-9,9), ""
-9 -2 -2 -7 7 9 -3 0 4 8 -3 -9 4 7 -7 8 -3 -4 8 -4

for (k = 1; k <= 20; ++k) print irand(0, 10^6), ""
986598 580968 627992 379949 700143 734615 361237
322631 116247 369376 509615 734421 321400 876989
940425 139472 255449 394759 113286 95688
```

Generating random integers in order

See Chapter 12 of Jon Bentley, *Programming Pearls*, 2nd ed., Addison-Wesley (2000), ISBN 0-201-65788-0. [ACM TOMS **6**(3), 359–364, September 1980].

```
% hoc
func bigrand() { return int(2^31 * rand()) }

# select(m,n): select m pseudo-random integers
# from (0,n) in order
proc select(m,n) \
{
    mleft = m
    remaining = n
    for (i = 0; i < n; ++i) \
    {
        if (int(bigrand() % remaining) < mleft) \
        {
            print i, ""
            mleft--
        }
        remaining--
    }
    println ""
}

select(3,10)
5 6 7
```

Generating random integers in order [cont.]

```
select(3,10)
0 7 8

select(3,10)
2 5 6

select(3,10)
1 5 7

select(10,100000)
7355 20672 23457 29273 33145 37562 72316 84442 88329 97929

select(10,100000)
401 8336 41917 43487 44793 56923 61443 90474 92112 92799

select(10,100000)
5604 8492 24707 31563 33047 41864 42299 65081 90102 97670
```

Testing pseudo-random number generators

Most tests are based on computing a χ^2 measure of computed and theoretical values. **If one gets values $p < 1\%$ or $p > 99\%$ for several tests, the generator is suspect.**

Marsaglia Diehard Battery test suite (1985): 15 tests. Marsaglia/Tsang `tuftest` suite (2002): 3 tests. All produce p values that can be checked for reasonableness.

These tests all expect *uniformly-distributed* pseudo-random numbers. How do you test a generator that produces pseudo-random numbers in some other distribution? You have to figure out a way to use those values to produce an expected uniform distribution that can be fed into the standard test programs. For example, take the negative log of exponentially-distributed values, since $-\log(\exp(-\text{random})) = \text{random}$. For normal distributions, consider successive pairs (x, y) as a 2-dimensional vector, and express in polar form (r, θ): θ is then uniformly distributed in $[0, 2\pi)$, and $\theta/(2\pi)$ is in $[0, 1)$.

Digression: The Birthday Paradox

The *birthday paradox* arises from the question *"How many people do you need in a room before the probability is at least half that two of them share a birthday?"*

The answer is just 23, not $365/2 = 182.5$.

The probability that *none* of n people are born on the same day is

$$\begin{aligned} P(1) &= 1 \\ P(n) &= P(n-1) \times (365 - (n-1))/365 \end{aligned}$$

The n-th person has a choice of $365 - (n - 1)$ days to not share a birthday with any of the previous ones. Thus, $(365 - (n - 1))/365$ is the probability that the n-th person is not born on the same day as any of the previous ones, assuming that they are born on different days.

Digression: The Birthday Paradox [cont.]

Here are the probabilities that n people *share* a birthday (i.e., $1 - P(n)$):

```
% hoc128
PREC = 3
p = 1
for (n = 1;n <= 365;++n) \
    {p *= (365-(n-1))/365; println n,1-p}
1 0
2 0.00274
3 0.00820
4 0.0164
...
22 0.476
23 0.507
24 0.538
...
100 0.999999693
...
```

$P(365) \approx 1.45 \times 10^{-157}$ [cf. 10^{80} particles in universe].

The Marsaglia/Tsang `tuftest` tests

❑ b’day test (generalization of Birthday Paradox).

❑ Euclid’s (ca. 330–225BC) gcd test.

❑ Gorilla test (generalization of monkey’s typing random streams of characters).

Euclid’s algorithm (ca. 300BC)

This is the oldest surviving nontrivial algorithm in mathematics.

```
func gcd(x,y) \
{ ## greatest common denominator of integer x, y
  r = abs(x) % abs(y)
  if (r == 0) return abs(y) else return gcd(y, r)
}

func lcm(x,y) \
{ ## least common multiple of integer x,y
  x = int(x)
  y = int(y)
  if ((x == 0) || (y == 0)) return (0)
  return ((x * y)/gcd(x,y))
}
```

Euclid’s algorithm (cont.)

Complete rigorous analysis of Euclid’s algorithm was not achieved until 1970–1990!

The average number of steps is

$$\begin{aligned} A\left(\mathsf{gcd}(x, y)\right) &\approx \Big((12\ln 2)/\pi^2\Big)\ln y \\ &\approx 1.9405\log_{10} y \end{aligned}$$

and the maximum number is

$$\begin{aligned} M\left(\mathsf{gcd}(x, y)\right) &= \lfloor \log_\phi\left((3-\phi)y\right)\rfloor \\ &\approx 4.785\log_{10} y + 0.6723 \end{aligned}$$

where $\phi = (1+\sqrt{5})/2 \approx 1.6180$ is the golden ratio.