A review of the review paper

Airy Processes and Variational Formulas

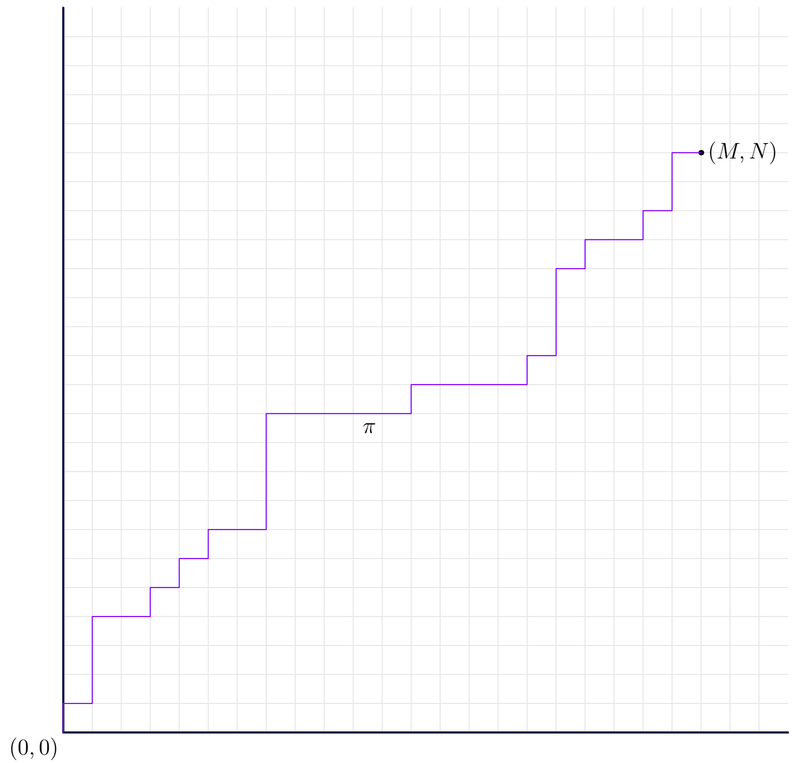
by Jeremy Quastel and Daniel Remenik

Airy Processes

- Fundamental (but new!) random processes on $\mathbb{R}$ (i.e. $x \mapsto \mathcal{A}(x)$ a random process)
- Are believed to govern the long time, large scale spatial fluctuations of random growth models in the *KPZ universality class*
- We will describe them through two models: *last passage percolation* and *the stochastic heat equation*

Last Passage Percolation

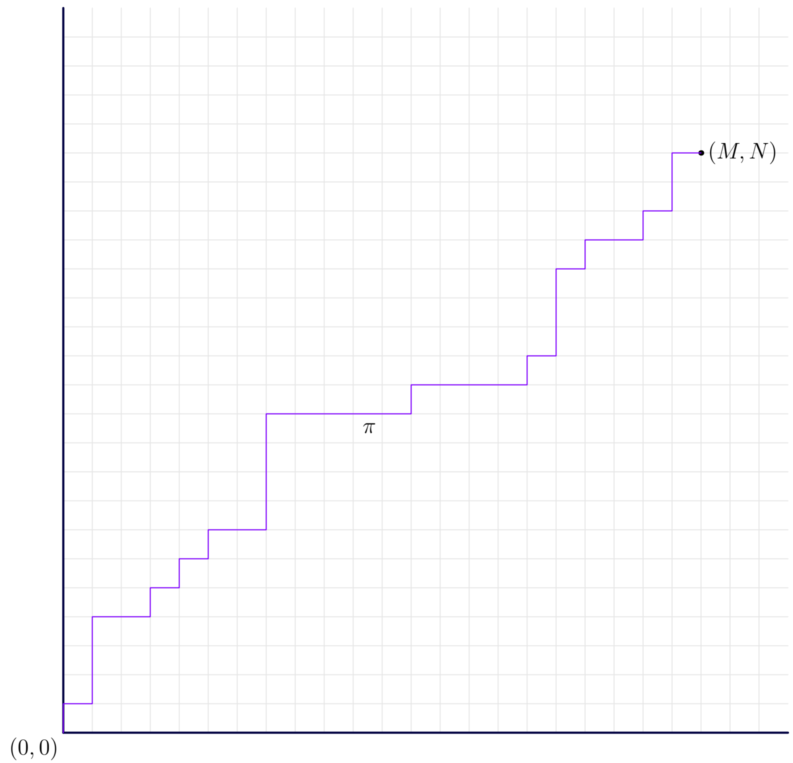
- Maximum of *correlated* random variables
- Input is iid random variables $\omega_{i,j}$ for $(i, j) \in \mathbb{N}^2$



$$L^{\text{point}}(M, N) := \max_{\pi:(0,0)\to(M,N)} \sum_{\mathbf{v}\in\pi} \omega_{\mathbf{v}}$$

Last Passage Percolation

- For $\omega_{i,j} \sim$ Geometric(q) the model becomes *integrable*
- Is a formula for the **exact** distribution of $L^{point}(M, N)$



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Last Passage Percolation

- For $\omega_{i,j} \sim$ Geometric(q) the model becomes *integrable*
- Is a formula for the **exact** distribution of $L^{point}(M, N)$
- Goes through the Robinson-Schensted-Knuth (RSK) bijection
- Deterministic, combinatorial bijection from arrays of non-negative integers to pairs of semi-standard Young tableaux with the same shape
- Length of top row of the shape is $L^{\mathrm{point}}(M, N)$

Last Passage Percolation

- For $\omega_{i,j} \sim$ Geometric(q) the induced measure on the shape is **Schur measure**
- Can study statistics of the shape using **Schur functions** (a special family of polynomials that are a basis for the space of symmetric polynomials of a given degree)
- Representation of Schur functions via determinants (Jacobi-Trudi identities) leads to representation of probabilities in terms of determinants

Last Passage Percolation

- Distribution of the passage time:

$$\mathbb{P}\left(L^{\text{point}}(M,N) \le n - N + 1\right) = \det(I-K)_{\ell^2\{n+1,n+2,\ldots\}}$$

where $K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ is given by

$$K(u,v) = \frac{\gamma_{N-1}}{\gamma_N} \frac{p_N(u)p_{N-1}(v) - p_{N-1}(u)p_N(v)}{u-v}\sqrt{w(u)w(v)}$$

with $w : \mathbb{N}_0 \to \mathbb{R}$ given by

$$w(k) = q^k \binom{k+M-N}{M-N}$$

and $p_N : \mathbb{N}_0 \to \mathbb{R}$ is the *degree* n *monic orthogonal polynomial* with respect to the weight w, and γ_N its L^2 norm

Last Passage Percolation

- Asymptotic statistics of the passage time:

$$\mu_q := 2\frac{\sqrt{q}+q}{1-q}, \quad \sigma_q := \frac{q^{1/6}(1+\sqrt{q})^{4/3}}{1-q}$$

- **Strong Law:**

$$\frac{1}{N}L^{\text{point}}(N,N) \xrightarrow[N\to\infty]{a.s.} \mu_q$$

- **Fluctuations:**

$$\frac{L^{\text{point}}(N,N)-\mu_q N}{\sigma_q N^{1/3}} \xrightarrow[N\to\infty]{(d)} \text{Tracy-Widom GUE distribution}$$

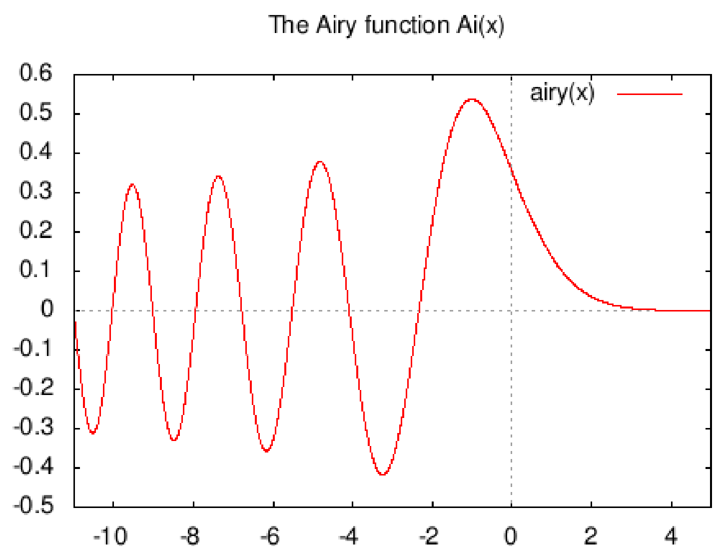
Tracy-Widom GUE Distribution

- Probability distribution with CDF

$$F_{\mathrm{GUE}}(s) = \det(I - P_s K_{\mathrm{Ai}} P_s)_{L^2(\mathbb{R})}$$

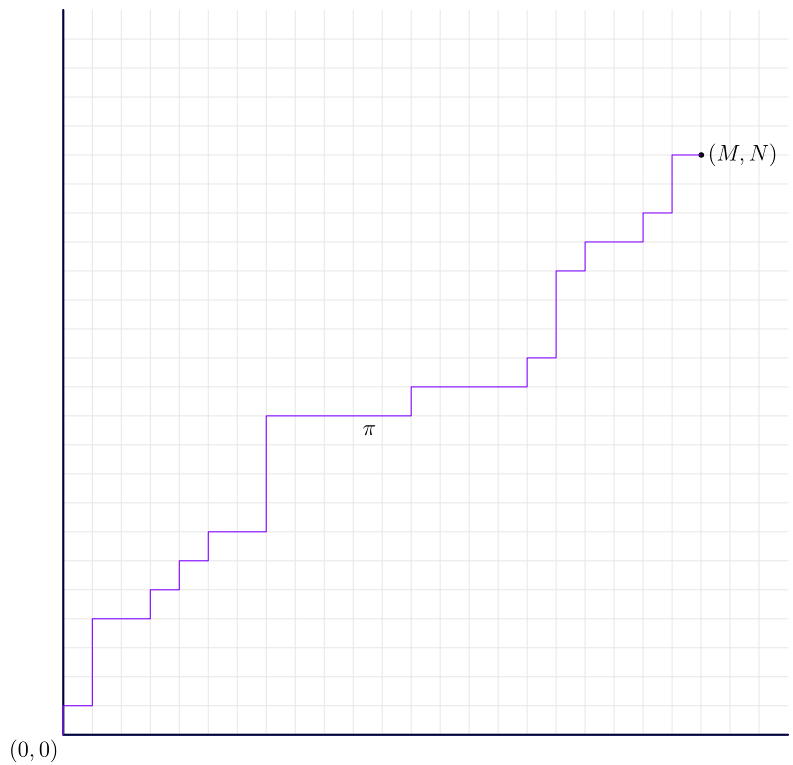
where P_s is projection onto the interval (s, ∞), and K_{Ai} is the Airy kernel (matrix)

$$K_{\mathrm{Ai}}(x, y) = \int_0^\infty \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)d\lambda$$



Last Passage Percolation

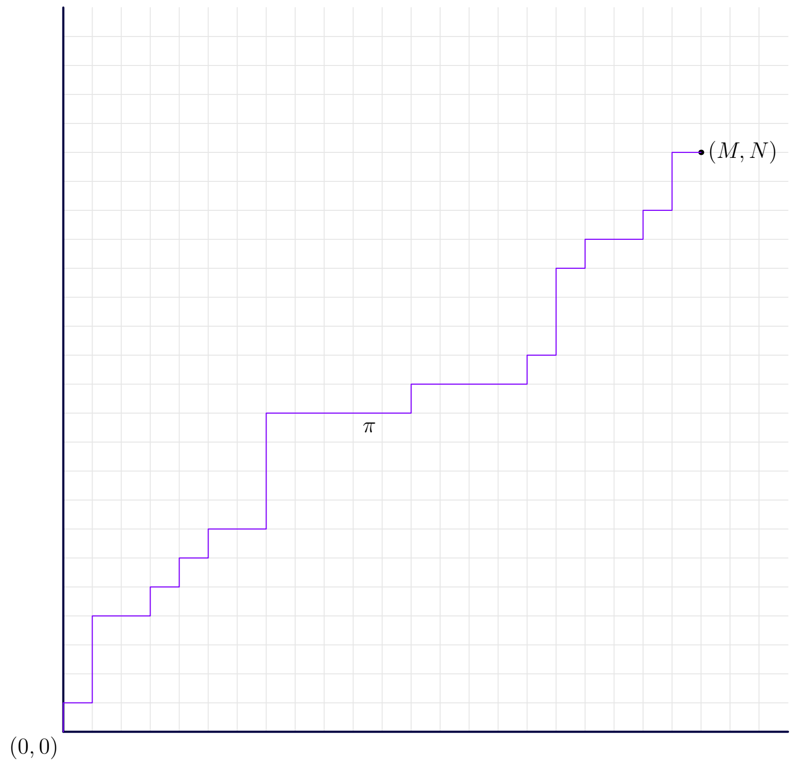
- Can turn this into a process by looking at the passage time at points “near” (N, N)



- How near is near?
- Turns out to be scale $N^{2/3}$ away from (N, N)

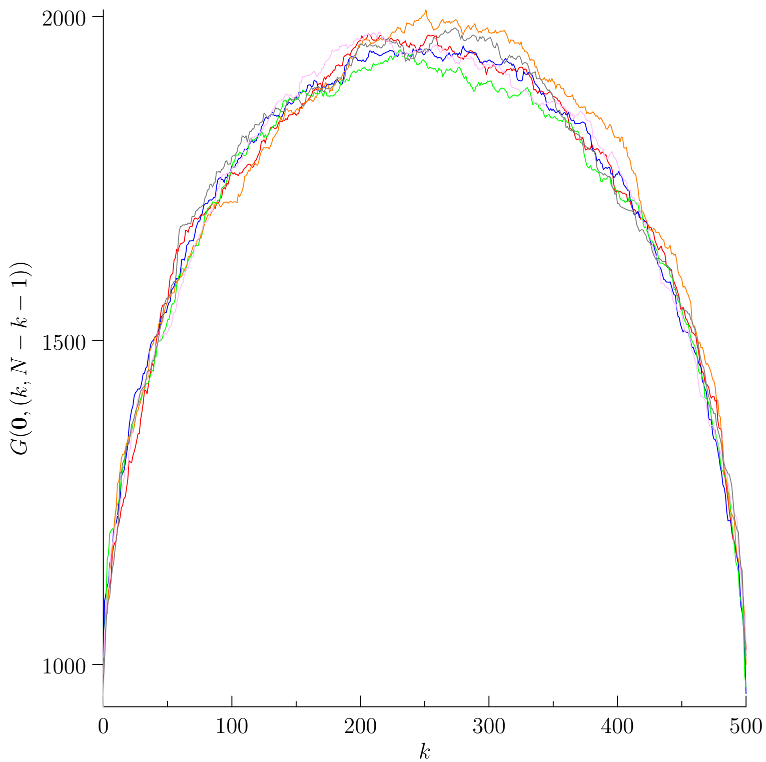
Last Passage Percolation

- Can turn this into a process by looking at the passage time at points "near" (N, N)



$$L^{\text{point}}(N+u, N-u) := c_1 N + c_2 N^{1/3} H_N^{\text{point}}(c_3 N^{-2/3} u)$$

Process of Passage Times



Last Passage Percolation

- **Theorem:** [Joh03]

$$H_N^{\text{point}}(u) \xrightarrow[N\to\infty]{(d)} \mathcal{A}_2(u) - u^2$$

as a process in u (in the topology of uniform convergence of continuous functions on compact sets)

- **Properties:**
  - the process $u \mapsto \mathcal{A}_2(u)$ is *stationary*
  - its one-point distributions are Tracy-Widom GUE
  - there is a formula (in fact several) for the multi-point distributions of $\mathcal{A}_2$, i.e.

$$\mathbb{P}\left(\mathcal{A}_2(u_1) \leq x_1, \ldots \mathcal{A}_2(u_n) \leq x_n\right)$$

Last Passage Percolation

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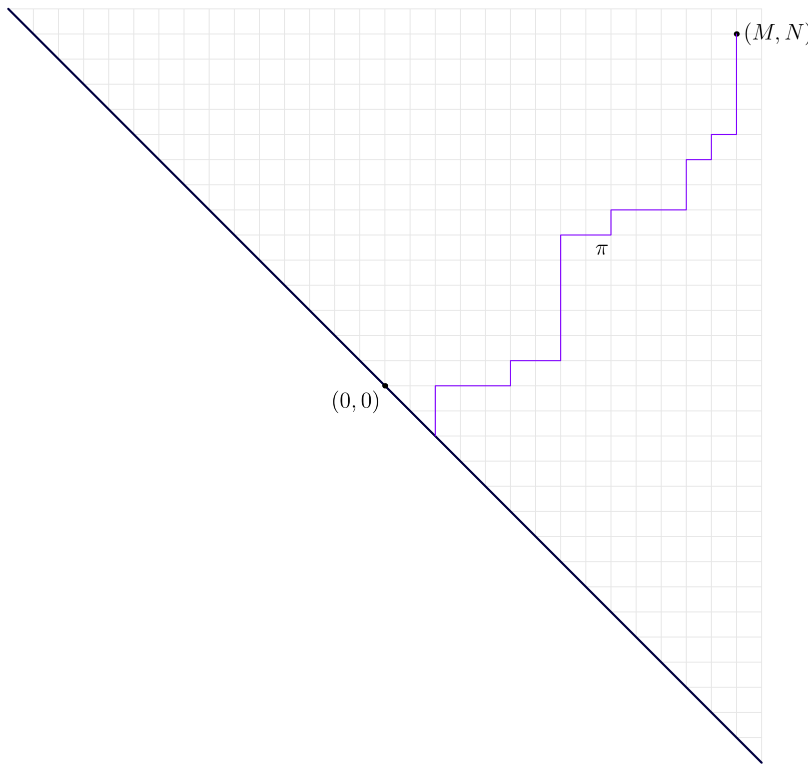
  as a process in u (in the topology of uniform convergence of continuous functions on compact sets)

- **Properties:**
  - the process $u \mapsto \mathcal{A}_2(u)$ is *stationary*
  - its one-point distributions are Tracy-Widom GUE
  - **Next time:** a formula for

$$\mathbb{P}\left(\mathcal{A}_2(u) \le g(u) \text{ for all } u \in [r, l]\right)$$

Last Passage Percolation

- Some variants of the passage times are integrable
- Can be described by modifying initial conditions



$$L^{\text{flat}}(M,N) := \max_{i \in \mathbb{Z}} \max_{\pi:(i,-i) \to (M,N)} \sum_{\mathbf{v} \in \pi} \omega_{\mathbf{v}}$$

Last Passage Percolation

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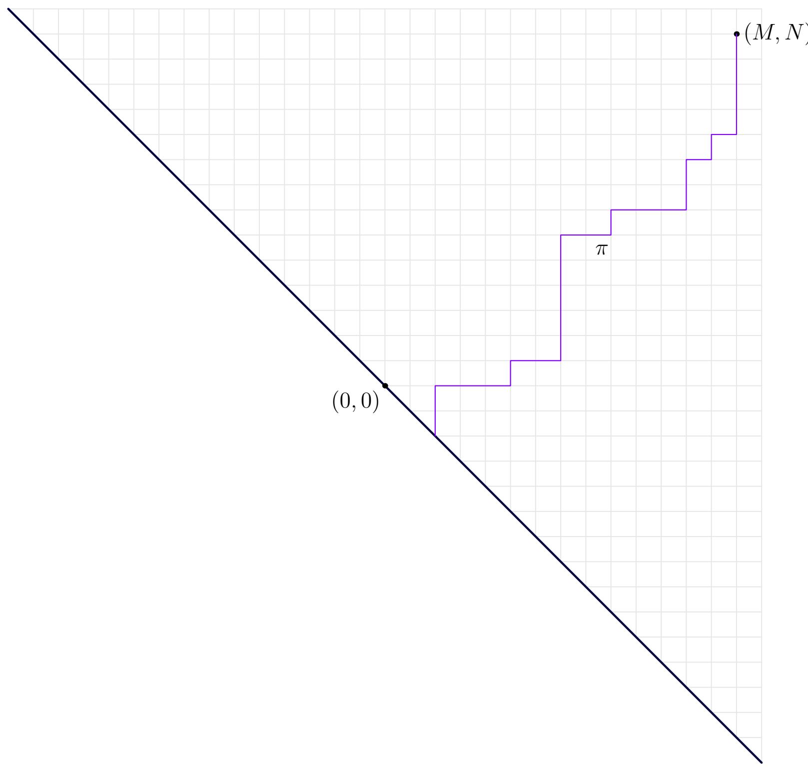


$$L^{\text{flat}}(N+u, N-u) := c_1 N + c_2 N^{1/3} H_N^{\text{line}}(c_3 N^{-2/3} u)$$

Last Passage Percolation

- **Theorem:** [Borodin-Ferrari-Pr"$a$hofer-Sasamoto]

$$H_N^{\text{line}}(u) \xrightarrow[N\to\infty]{(d)} \mathcal{A}_1(u)$$

in the sense of convergence of finite-dimensional distributions

- **Properties:**

  - the process $u \mapsto \mathcal{A}_1(u)$ is *stationary*

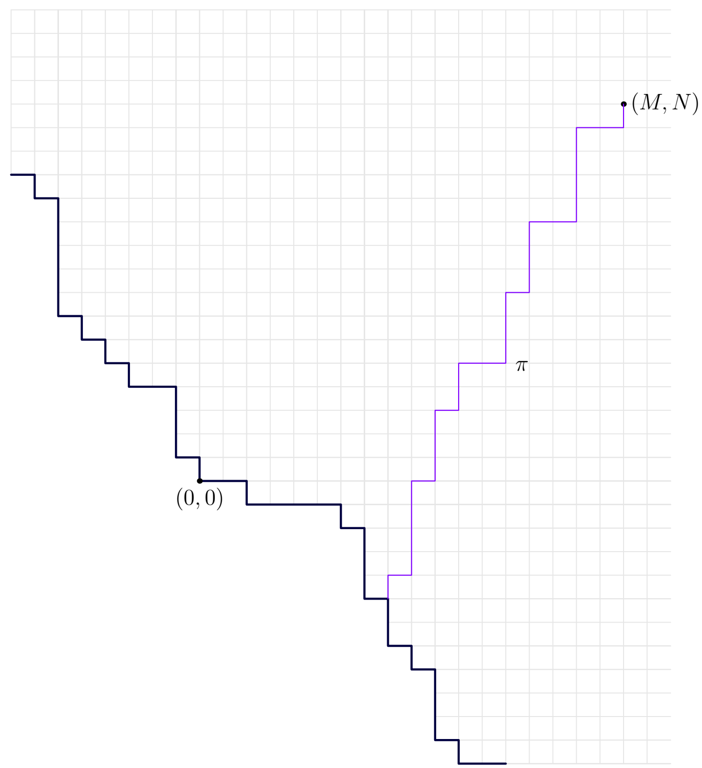
  - its one-point distributions are Tracy-Widom GOE

  - there is a formula (in fact several) for the multi-point distributions of $\mathcal{A}_1$, i.e.

$$\mathbb{P}\left(\mathcal{A}_1(u_1) \le x_1, \dots \mathcal{A}_1(u_n) \le x_n\right)$$

Last Passage Percolation

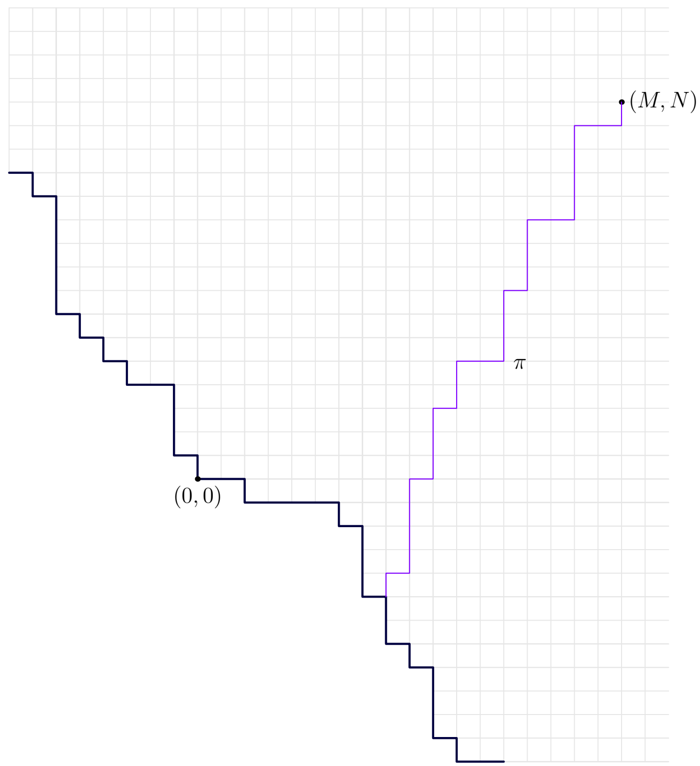
- Some variants of the passage times are integrable
- Stationary version: boundary a 2-sided random walk



$$L^{\mathrm{stat}}(M,N) := \max_{\mathbf{u}\in S} \max_{\pi:\mathbf{u}\to(M,N)} \sum_{\mathbf{v}\in\pi} \omega_{\mathbf{v}}$$

Last Passage Percolation

- Some variants of the passage times are integrable
- Stationary version: boundary a 2-sided random walk



$$L^{\mathrm{stat}}(N+u, N-u) := c_1 N + c_2 N^{1/3} H_N^{\mathrm{stat}}(c_3 N^{-2/3} u)$$

Last Passage Percolation

- **Theorem:** [Borodin-Ferrari-Pr"$a$hofer]

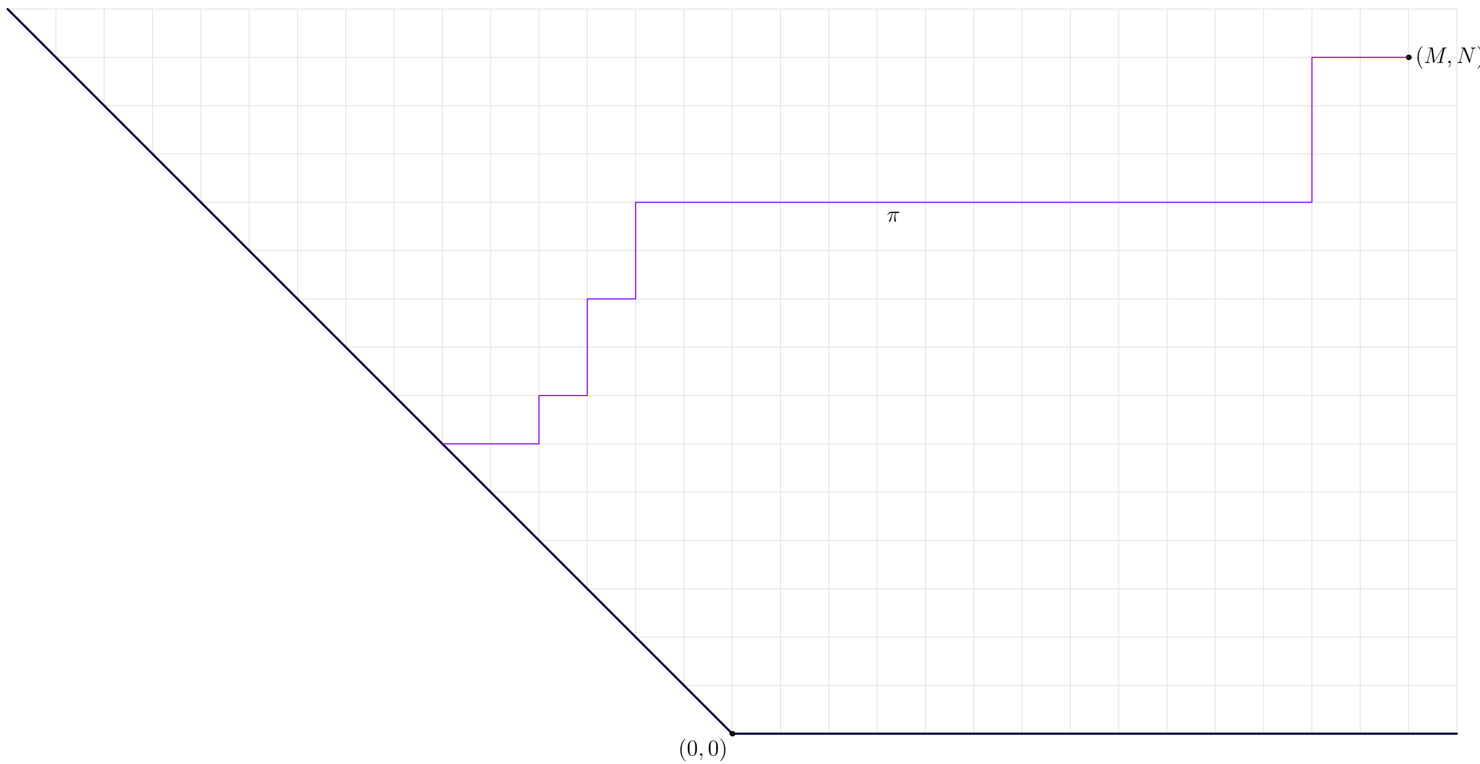
$$H_N^{\rm stat}(u) \xrightarrow[N\to\infty]{(d)} \mathcal{A}_{\rm stat}(u)$$

in the sense of convergence of finite-dimensional distributions

- **Properties:**
  - the process $u \mapsto \mathcal{A}_1(u)$ is **not** stationary
  - $\mathcal{A}_{\rm stat}$ is a double-sided Brownian motion with a random height shift at the origin

Last Passage Percolation

- Can also mix boundary conditions
- The corner-flat process



$$L^{\text{half-line}}(N+u, N-u) = c_1 N + c_2 N^{1/3} H_N^{\text{half-line}}(c_3 N^{-2/3} u)$$

Last Passage Percolation

- **Theorem:** [Borodin-Ferrari-Sasamoto]

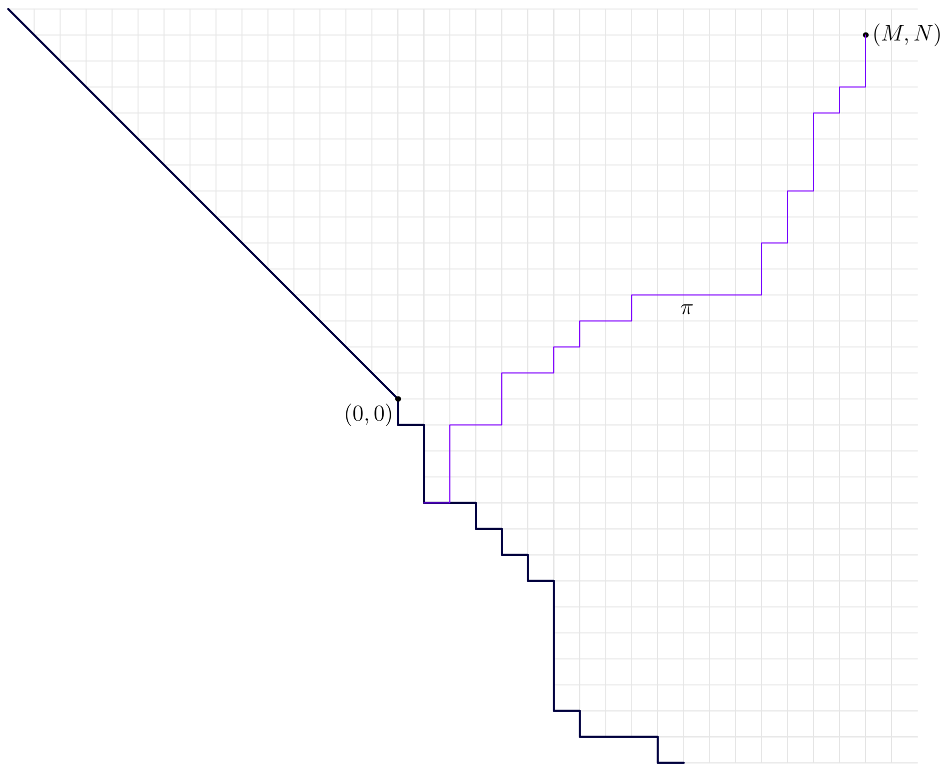
$$H_N^{\text{half-line}}(u) - u^2 \mathbf{1}\{u \geq 0\} \xrightarrow[N\to\infty]{(d)} \mathcal{A}_{2\to 1}(u)$$

  in the sense of convergence of finite-dimensional distributions

- **Properties:**
  - the process $u \mapsto \mathcal{A}_{2\to 1}(u)$ is **not** stationary
  - one-point distribution of $\mathcal{A}_{2\to 1}(x)$ is known
  - $\mathcal{A}_{2\to 1}(u+v) \to \mathcal{A}_2(u)$ as $v \to \infty$
  - $\mathcal{A}_{2\to 1}(u+v) \to 2^{1/3}\mathcal{A}_1(2^{-2/3}u)$ as $v \to -\infty$

Last Passage Percolation

- Can also mix boundary conditions
- The flat-stat process



$$L^{\text{flat-stat}}(N+u, N-u) = c_1 N + c_2 N^{1/3} H_N^{\text{flat-stat}}(c_3 N^{-2/3} u)$$

Last Passage Percolation

- **Theorem:** (I think)

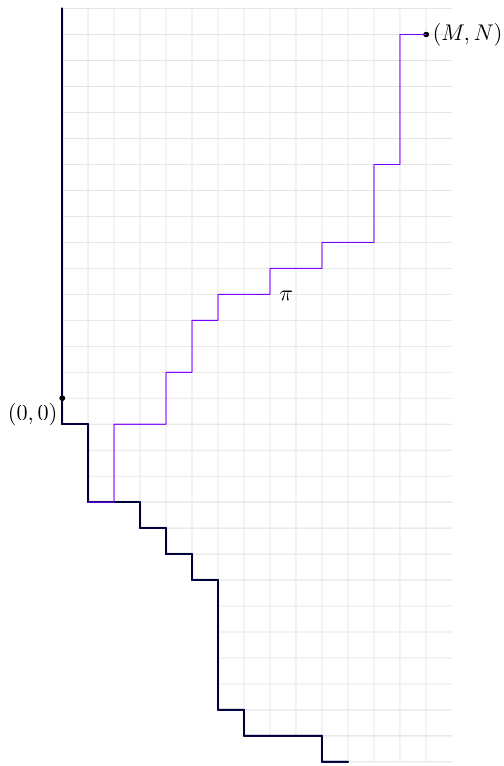
$$H_N^{\text{flat-stat}}(u) \xrightarrow[N\to\infty]{(d)} \mathcal{A}_{1\to\text{BM}}(u)$$

  in the sense of convergence of finite-dimensional distributions

- **Properties:**
  - the process $u \mapsto \mathcal{A}_{1\to\text{BM}}(u)$ is **not** stationary
  - one-point distribution of $\mathcal{A}_{1\to\text{BM}}(x)$ is known
  - $\mathcal{A}_{1\to\text{BM}}(u+v) \to \mathcal{A}_{\text{stat}}(u)$ as $v \to \infty$
  - $\mathcal{A}_{1\to\text{BM}}(u+v) \to \mathcal{A}_1(u)$ as $v \to -\infty$

Last Passage Percolation

- Can also mix boundary conditions
- The corner-stat process



$$L^{\text{corner-stat}}(N+u, N-u) = c_1 N + c_2 N^{1/3} H_N^{\text{corner-stat}}(c_3 N^{-2/3} u)$$

Last Passage Percolation

- **Theorem:** (I think)

$$H_N^{\text{corner-stat}}(u) - u^2 \mathbf{1}\{u \le 0\} \xrightarrow[N\to\infty]{(d)} \mathcal{A}_{2\to\text{BM}}(u)$$

  in the sense of convergence of finite-dimensional distributions

- **Properties:**
  - the process $u \mapsto \mathcal{A}_{2\to\text{BM}}(u)$ is **not** stationary
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  - $\mathcal{A}_{2\to\text{BM}}(u+v) \to \mathcal{A}_{\text{stat}}(u)$ as $v \to \infty$
  - $\mathcal{A}_{2\to\text{BM}}(u+v) \to \mathcal{A}_{2}(u)$ as $v \to -\infty$

Airy Processes

- In a sense, the $\mathcal{A}_2$ process is the "fundamental" process and all other processes can be derived from it (at least their one-point distributions)
- Follows because general initial conditions are a superposition of corner initial conditions
- Example:

$$L^{\text{line}}(N) = \max_{k=-N,\dots,N} L^{\text{point}}(N+k, N-k)$$

Translated into scaling limits this becomes

$$\mathcal{A}_1(x) = \sup_{x\in\mathbb{R}} \left\{\mathcal{A}_2(x) - x^2\right\}$$

Stochastic Heat Equation

- Variational formula relation between different processes can be understood in terms of the *stochastic heat equation*

$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + WZ, \quad Z(0,x) = \delta_0(x)$$

where W is *space-time white noise*

- **Without** the noise term the solution is

$$\varrho(t,x) = e^{-x^2/2t}\sqrt{2\pi t}$$

Stochastic Heat Equation

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$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + WZ, \quad Z(0,x) = \delta_0(x)$$

where W is *space-time white noise*

- **With** the noise term the solution is $\varrho(t,x)$ times a stationary process

$$Z(t,x) = \varrho(t,x)e^{-t/24}\exp\left\{t^{1/3}A_t(t^{-2/3}x)\right\}$$

- **Conjecture:** $A_t(x) \to \mathcal{A}_2(x)$ as $t \to \infty$, on the process level
- Known for the one-dimensional distributions

Stochastic Heat Equation

- Now use that solution is linear in the initial data

$$\partial_t Z_{\mathrm{f}} = \frac{1}{2}\partial_{xx} Z_{\mathrm{f}} + W Z_{\mathrm{f}}, \qquad Z_{\mathrm{f}}(0,x) = 1$$

- On the one hand, we can define

$$Z_{\mathrm{f}}(t,x) = e^{-t/24} \exp\left\{ t^{1/3} A_t^{\mathrm{flat}}(t^{-2/3}x) \right\}$$

and we expect that $A_t^{\mathrm{flat}}(x) \to \mathcal{A}_1(x)$. On the other hand

$$Z_{\mathrm{f}}(t,x) =: \int \varrho(t, x-y) e^{-t/24} \exp\left\{ t^{1/3} A_t(t^{-2/3}(x-y)) \right\} dy$$

- Steepest descent type heuristic suggests

$$t^{1/3} A_t^{\mathrm{flat}}(t^{-2/3}x) \sim \sup_{y\in\mathbb{R}} \left\{ \log \varrho(t, x-y) + t^{1/3} A_t(t^{-2/3}(x-y)) \right\}$$

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- Using statistical scaling properties one gets

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- Using statistical scaling properties one gets

$$A_t^{\mathrm{flat}}(x) \sim \sup_{y \in \mathbb{R}} \left\{ -(x-y)^2 + A_t(x-y) \right\}$$

Stochastic Heat Equation

- Now use that solution is linear in the initial data

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- Now taking $t \to \infty$ one gets

$$\mathcal{A}_1(x) \stackrel{(d)}{=} \sup_{y\in\mathbb{R}} \left\{ -(x-y)^2 + \mathcal{A}_2(x-y) \right\}$$

Hitting Probabilities for $\mathcal{A}_2$

$$\mathbb{P}\left(\mathcal{A}_2(u) \leq g(u) \text{ for all } u \in [l, r]\right)$$

- Simplifications:
  - can take $[l, r] = [0, r - l] = [0, T]$ by stationarity of $\mathcal{A}_2$
  - can replace g with $\hat{g}(t) = g(l + r - t)$ by invariance of $\mathcal{A}_2$ under $u \mapsto -u$
- There are formulas for

$$\mathbb{P}(\mathcal{A}_2(u_1) \leq x_1, \ldots, \mathcal{A}_2(u_n) \leq x_n)$$

expressed in terms of Fredholm determinants

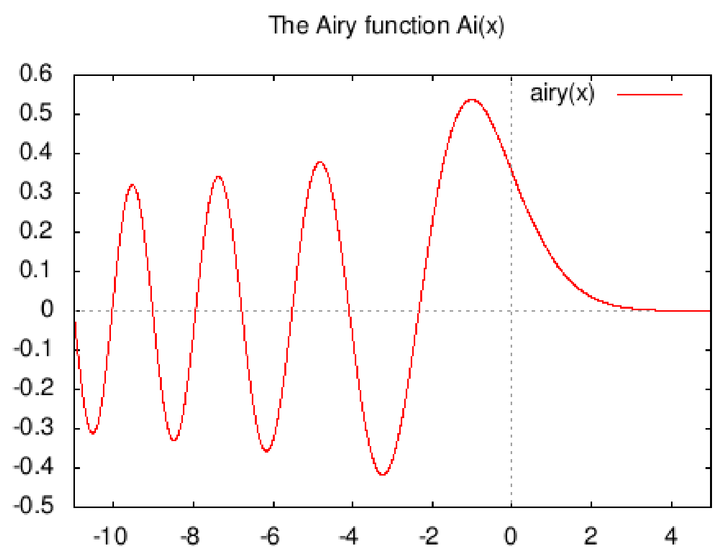
Fredholm Determinants

- Recall the Tracy-Widom GUE distribution has CDF

$$F_{\mathrm{GUE}}(s) = \det(I - P_s K_{\mathrm{Ai}} P_s)_{L^2(\mathbb{R})}$$

where P_s is projection onto the interval (s, ∞), and K_{Ai} is the Airy kernel (matrix)

$$K_{\mathrm{Ai}}(x, y) = \int_0^\infty \mathrm{Ai}(x + \lambda)\mathrm{Ai}(y + \lambda)d\lambda$$



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where P_s is projection onto the interval (s, ∞), and K_{Ai} is the Airy kernel (matrix)

$$K_{\mathrm{Ai}}(x, y) = \int_0^\infty \mathrm{Ai}(x + \lambda)\mathrm{Ai}(y + \lambda) d\lambda$$

- $\mathrm{Ai}(x)$ solves $H\mathrm{Ai} = 0$, where $H = -\partial_x^2 + x$ is the *Airy Hamiltonian*, boundary condition is $\mathrm{Ai}(x) \to 0$ as $x \to \infty$

$$H\mathrm{Ai}(\cdot + \lambda) = -\lambda \mathrm{Ai}(\cdot + \lambda)$$

so K_{Ai} is projection onto the negative eigenspace of H

Fredholm Determinants

- Recall the Tracy-Widom GUE distribution has CDF

$$F_{\mathrm{GUE}}(s) = \det(I - P_s K_{\mathrm{Ai}} P_s)_{L^2(\mathbb{R})}$$

where P_s is projection onto the interval (s, ∞), and K_{Ai} is the Airy kernel (matrix)

$$K_{\mathrm{Ai}}(x, y) = \int_0^\infty \mathrm{Ai}(x + \lambda)\mathrm{Ai}(y + \lambda)d\lambda$$

- Also, think of everything as matrices
  - K_{Ai} is a matrix determined by the Airy function
  - $P_s K_{\mathrm{Ai}} P_s$ is the lower $[s, \infty) \times [s, \infty)$ block of the K_{Ai} matrix, zero elsewhere

Fredholm Determinants

- OK fine, it's a matrix, so what does $\det$ mean?
- $\mathrm{Defn:}$ If K is an integral operator on $L^2(X, d\mu)$ with kernel $K(x,y)$, i.e.

$$(Kf)(x) = \int K(x,y) f(y)\, d\mu(y)$$

then by definition

$$\det(I + \lambda K) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_X \dots \int_X \det[K(x_i, x_j)]_{i,j=1}^n \, d\mu(x_1) \dots \, d\mu(x_n)$$

- Why?
- It holds in finite dimensions too!

Fredholm Determinants

- Let K be an $n \times n$ matrix

$$\lambda \mapsto \det(I + \lambda K) = \sum_{k=0}^{n} a_k \lambda^k$$

  is a degree n polynomial in λ

- $a_k = \frac{1}{k!} \partial_\lambda^k \det(I + \lambda K)$
- Use that the determinant is a linear function of each of its columns, so if $M(\lambda) = [M_1(\lambda)|M_2(\lambda)|\ldots|M_n(\lambda)]$ then

$$\begin{aligned}
\det M(\lambda+\epsilon) &= \det[M_1(\lambda) + \epsilon\partial_\lambda M_1(\lambda)|\ldots|M_n(\lambda) + \epsilon\partial_\lambda M_n(\lambda)] + O(\epsilon^2) \\
&= \det M(\lambda) + \epsilon \sum_{k=1}^{n} \det[M_1(\lambda)|\ldots|\partial_\lambda M_k(\lambda)|\ldots|M_n(\lambda)]
\end{aligned}$$

Fredholm Determinants

- Let K be an $n \times n$ matrix

$$\lambda \mapsto \det(I + \lambda K) = \sum_{k=0}^{n} a_k \lambda^k$$

is a degree n polynomial in λ

- $a_k = \frac{1}{k!} \partial_\lambda^k \det(I + \lambda K)$
- Use that the determinant is a linear function of each of its columns, so if $M(\lambda) = [M_1(\lambda)|M_2(\lambda)|\dots|M_n(\lambda)]$ then

$$\partial_\lambda \det M(\lambda) = \sum_{k=1}^{n} \det[M_1(\lambda)|\dots|\partial_\lambda M_k(\lambda)|\dots|M_n(\lambda)]$$

$$\partial_\lambda (I + \lambda K)_k = K_k$$

Fredholm Determinants: Simple Example

$$\det(I + \lambda K) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_X \dots \int_X \det[K(x_i, x_j)]_{i,j=1}^n \, d\mu(x_1) \dots \, d\mu(x_n)$$

- Take $K(x, y) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ on $L^2(\mathbb{R})$

$$\det(I - P_s K P_s) = \int_{-\infty}^{s} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$

Multipoint Distributions for $\mathcal{A}_2$

$$\mathbb{P}(\mathcal{A}_2(t_1) \le x_1, \dots \mathcal{A}_2(t_n) \le x_n) = \det(I - f^{1/2} K_{\mathrm{Ai}}^{\mathrm{ext}} f^{1/2})_{L^2(\{t_1,\dots,t_n\}\times\mathbb{R})}$$

with $f(t_j, x) = \mathbf{1}\{x \ge t_j\}$ and $K_{\mathrm{Ai}}^{\mathrm{ext}}$ the *extended Airy kernel*

- Another formula that is more useful for us

$$\det(I - K_{\mathrm{Ai}} + \bar{P}_{x_1} e^{(t_1-t_2)H} \bar{P}_{x_2} e^{(t_2-t_3)H} \dots \bar{P}_{x_n} e^{(t_n-t_1)H} K_{\mathrm{Ai}})_{L^2(\mathbb{R})}$$

with $\bar{P}_a f(x) = \mathbf{1}\{x \le a\} f(x)$, and $H = -\partial_x^2 + x$

- Want a formula for $\mathbb{P}\left(\mathcal{A}_2(u) \le g(u) \text{ for all } u \in [0,T]\right)$
- Take a fine mesh $t_k = k2^{-n}T$ with $k = 0, 1, \dots, 2^n$, then take a limit of the above formula as $n \to \infty$
- Clearly easier to do this with the second formula

Multipoint Distributions for $\mathcal{A}_2$

- Another formula that is more useful for us

$$\det(I - K_{\mathrm{Ai}} + \bar{P}_{x_1} e^{(t_1-t_2)H} \bar{P}_{x_2} e^{(t_2-t_3)H} \dots \bar{P}_{x_n} e^{TH} K_{\mathrm{Ai}})_{L^2(\mathbb{R})}$$

with $\bar{P}_a f(x) = \mathbf{1}\{x \le a\} f(x)$, and $H = -\partial_x^2 + x$

- Want a formula for $\mathbb{P}\left(\mathcal{A}_2(u) \le g(u) \text{ for all } u \in [0,T]\right)$
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- Clearly easier to do this with the second formula
- Take a limit of the operator

$$\bar{P}_{g(t_1)} e^{(t_1-t_2)H} \bar{P}_{g(t_2)} e^{(t_2-t_3)H} \dots e^{(t_{n-1}-t_n)H} \bar{P}_{g(t_n)}$$

- Take a limit of the operator

$$\bar{P}_{g(t_1)} e^{(t_1 - t_2)H} \bar{P}_{g(t_2)} e^{(t_2 - t_3)H} \dots e^{(t_{n-1} - t_n)H} \bar{P}_{g(t_n)}$$

- For $t > 0$, if we define $u(t, x) = (e^{-tH} f)(x)$ then u solves

$$\partial_t u = -Hu = \partial_x^2 u - xu, \quad u(0, x) = f(x)$$

- So if we apply this operator to a function,
  - it kills off the function above $g(t_n)$,
  - puts that in as an initial condition and solves a heat equation to time $t_n - t_{n-1}$,
  - then kills off the solution above $g(t_{n-1})$
  - puts that in as an initial condition and solves a heat equation to time $t_{n-1} - t_{n-2}$,
  - ... (now simplify and replace g with $\hat{g}$)

- Take a limit of the operator

$$\bar{P}_{g(t_1)} e^{(t_1-t_2)H} \bar{P}_{g(t_2)} e^{(t_2-t_3)H} \ldots e^{(t_{n-1}-t_n)H} \bar{P}_{g(t_n)}$$

- For $t > 0$, if we define $u(t,x) = (e^{-tH} f)(x)$ then u solves

$$\partial_t u = -Hu = \partial_x^2 u - xu, \quad u(0,x) = f(x)$$

- Hence if we let $u(t,x)$ be the solution to

$$\partial_t = -Hu \text{ for } x < g(t), \quad \begin{aligned} u(0,x) &= f(x)\mathbf{1}\{x < g(0)\}, \\ u(t,x) &= 0 \text{ for } x \geq g(t) \end{aligned}$$

and let Θ_T^g be the operator which takes $f(\cdot)$ to $u(T,\cdot)$, then

$$\bar{P}_{g(t_1)} e^{(t_1-t_2)H} \bar{P}_{g(t_2)} e^{(t_2-t_3)H} \ldots e^{(t_{n-1}-t_n)H} \bar{P}_{g(t_n)} \to \Theta_T^g$$

Hitting Probabilities for $\mathcal{A}_2$

$$\mathbb{P}\ (\mathcal{A}_2(t_1) \le g(t_1), \ldots \mathcal{A}_2(t_n) \le g(t_n))$$
$$= \det(I - K_{\mathrm{Ai}} + \bar{P}_{g(t_n)} e^{(t_1-t_2)H} \bar{P}_{g(t_{n-1})} e^{(t_2-t_3)H} \ldots \bar{P}_{g(t_1)} e^{(t_n-t_1)H} K_{\mathrm{Ai}})_{L^2(\mathbb{R})}$$

- Hence we conclude that

$$\mathbb{P}\,(\mathcal{A}_2(u) \le g(u) \text{ for all } u \in [0,T]) = \det(I - K_{\mathrm{Ai}} + \Theta_T^g e^{TH} K_{\mathrm{Ai}})$$

- Θ_T^g has an integral kernel and it can be computed, i.e.

$$u(T,x) = (\Theta_T^g f)(x) = \int \Theta_T^g(x,y) f(y)\, dy$$

and there is an explicit formula for $\Theta_T^g(x,y)$

Hitting Probabilities for $\mathcal{A}_2$

$$\mathbb{P}\ (\mathcal{A}_2(t_1) \le g(t_1), \ldots \mathcal{A}_2(t_n) \le g(t_n))$$
$$= \det(I - K_{\mathrm{Ai}} + \bar{P}_{g(t_n)} e^{(t_1-t_2)H} \bar{P}_{g(t_{n-1})} e^{(t_2-t_3)H} \ldots \bar{P}_{g(t_1)} e^{(t_n-t_1)H} K_{\mathrm{Ai}})_{L^2(\mathbb{R})}$$

- Hence we conclude that

$$\mathbb{P}\left(\mathcal{A}_2(u) \le g(u) \text{ for all } u \in [0,T]\right) = \det(I - K_{\mathrm{Ai}} + \Theta_T^g e^{TH} K_{\mathrm{Ai}})$$

- Θ_T^g has an integral kernel and it can be computed

$$\Theta_T^g(x,y) = e^{-Ty+T^3/3} \varrho_T(x,y) \mathbb{P}_{\hat{B}(0)=x, \hat{B}(T)=y-T^2}\left(\hat{B}(s) \le g(s) - s^2 \text{ on } [0,T]\right)$$

where $\hat{B}$ is a Brownian bridge

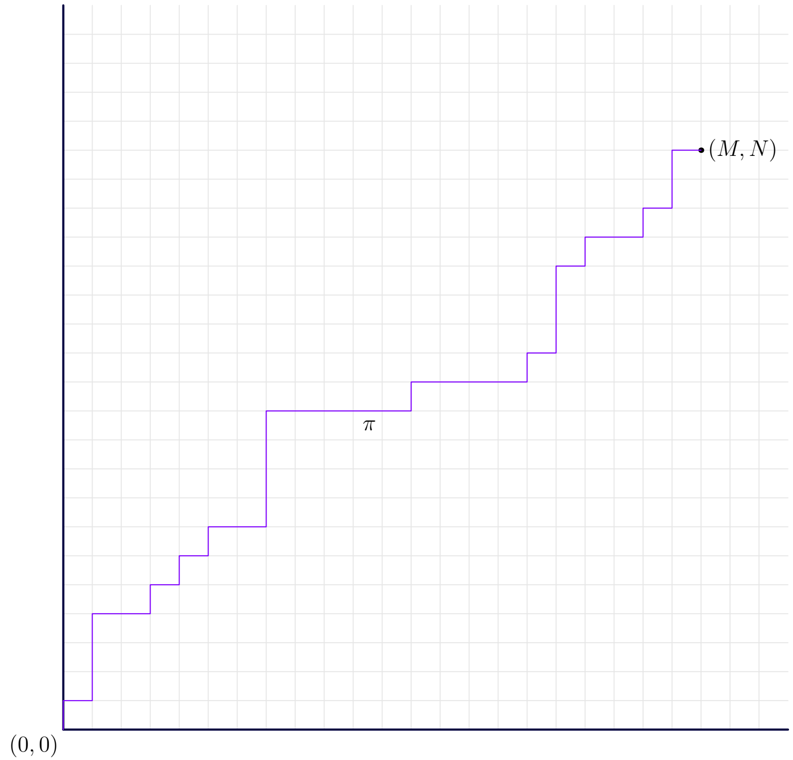
- Simplest case is clearly $g(t) = m + t^2$ for $m > 0$

Hitting Probabilities for $\mathcal{A}_2$

$$\mathbb{P}(\mathcal{A}_2(t) \le m + t^2 \text{ for } -L \le t \le L) = \det(I - K_{\mathrm{Ai}} + \Theta_L e^{2LH} K_{\mathrm{Ai}})$$

with $\Theta_L = \Theta_{[-L,L]}^{g(t)=t^2+m}$

Endpoint Distribution for Geometric LPP



$$L^{\text{point}}(M,N) := \max_{\pi:(0,0)\to(M,N)} \sum_{\mathbf{v}\in\pi} \omega_{\mathbf{v}}$$

- Let $\kappa_N = \text{argmax}_{k=-N\ldots N} L^{\text{point}}(N+k, N-k)$ (rightmost point)

- Then $c_3 N^{-1} \kappa_N \xrightarrow{(d)} \mathrm{argmax}_{t \in \mathbb{R}} \{\mathcal{A}_2(t) - t^2\}$

Hitting Probabilities for $\mathcal{A}_2$

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