Odd perfect numbers

Final report
Summer Independent REU

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SOME DEFINITIONS

DIVISOR SIGMA $\sigma(n)$

Let the factorization of a number n be represented by $n = p_1^{a_1} p_2^{a_2} \ldots p_m^{a_m}$ where each p is a distinct prime. For this paper we define the divisor sigma as

$$\sigma(n) \equiv \prod_{i=1}^{m} \sum_{j=0}^{a_i} p_i^j \ . \tag{1}$$

Notice that the divisor sigma is equivalent to the sum of the integer divisors of n including 1 and n itself. The Sigma function is a multiplicative function which means that $\sigma(ab) = \sigma(a)\sigma(b)$ when a and b share no common factors.

PERFECT NUMBER

A perfect number is one that satisfies the relation

$$\sigma(n) = 2n \ . \tag{2}$$

LEAST PRIME Y

Y is the smallest prime number in the factorization of n.

CO-PRIME RESIDUAL OF n, k

k is the integer such that $n = Y^a k$.

ODD PERFECT NUMBER

An odd perfect number is an n that satisfies (2) and is such that $Y \neq 2$.

We note that

$$\frac{2k}{\sigma(k)} = Y^{-a} \sigma\left(Y^a\right) \tag{3}$$

And because y is of the form of the power of a prime its sigma is of the form $\sigma\left(y^a\right) = (1 + y + \ldots + y^a)$ allowing us to simplify equation (3) to

(4) $$\frac{2k}{\sigma(k)} = \sum_{i=0}^{a} Y^{-i} .$$

We can limit the range of equation 4 by noting that.

(5) $$1 < \frac{2k}{\sigma(k)} < \frac{1}{1-Y^{-1}} .$$

If k is prime and Y>2 then $k \geq 5$ and $2k/\sigma(k) = 2k/(k+1)$ monotonically approaches 2. Therefore (2)(10)/(5+1)=10/6 forms a lower bound on the left hand side of equation (4) but if the ratio was 10/6 then the inequality (5) would be violated since 3/2 is the maximum the ratio can be for odd perfects and $(1-1/3)^{-1} = 3/2 < 10/6$ Therefore there can be no odd perfect numbers with k prime.

Now let us consider a number whose form is $n = y^{a_1} x^b$ we know that.

$\dfrac{2x^b}{\left(\sum_{i=0}^{b} x^i\right)} = \dfrac{2x^b}{\left(\dfrac{x^{b+1}-1}{x-1}\right)} < 3/2$ if n is an odd perfect number. To find the minimum value of this expression where $x \geq 5$ we take its derivative.

$$\frac{d}{dx}\left(\frac{2x^b}{\left(\frac{x^{b+1}-1}{x-1}\right)}\right) = \frac{d}{dx}\left(\frac{2\left(x^{b+1}-x^b\right)}{x^{b+1}-1}\right) = \frac{\left(2(b+1)x^b - 2bx^{b-1}\right)}{x^{b+1}-1} + \frac{-1(b+1)x^b}{\left(x^{b+1}-1\right)^2}\left(2\left(x^{b+1}-x^b\right)\right)$$

which simplifies to $\dfrac{2x^b\left(x^{b+1}-x(b+1)+b\right)}{x\left(x^{b+1}-1\right)^2}$.

Which is positive when $x^{b+1} + b > (b+1)x$, which is true for all x>1.

Therefore we can conclude that the ratio $2k/(k)$ is monotonically increasing with increasing x so its minimum is at the endpoint where x is least. To show that there can be no odd perfect numbers of this form then we need only show that

(6) $$\frac{8(5^b)}{5^{b+1}-1} > 3/2$$

for all b>1. We simply note that $\frac{8(5^b)}{5^{b+1}-1} = \frac{8}{5-1/5^b} > \frac{8}{5} > \frac{3}{2}$. We conclude that there can be no odd perfect numbers composed only of the powers of two unique primes.

In general we can factor out all but one prime yielding

(7) $$Y^{(-a_1)}\sigma(Y)k^{-1}x_{\mathrm{m}}^{(a_m)} = 2x_{\mathrm{m}}^{(a_{\mathrm{m}})} / \sigma(k)$$

Since sigma is multiplicative we can split the right hand side of (7) into $\frac{2x_m^{a_m}}{\sigma(x_m^{a_m})}\sigma\left(\frac{k}{x_m^{a_m}}\right)^{-1}$.

We know that k cannot be abundant because if it were then it would also violate inequality (5) since the ratio would be less than 1. Thus we can express the inequalities (8) and (9).

(8) $$\frac{3}{2}k^{-1}x_m^{a_m} < Y^{-a_1}\sigma(Y)k^{-1}x_m^{a_m}$$

(9) $$\frac{8}{5}\sigma\left(\frac{k}{x_m^{a_m}}\right)^{-1} > \frac{2x_m^{a_m}}{\sigma(x_m^{a_m})}\sigma\left(\frac{k}{x_m^{a_m}}\right)^{-1}$$

We can see that $\frac{3}{2}k^{-1}x_m^{a_m} < \dfrac{8}{5\sigma\left(\dfrac{k}{x_m^{a_m}}\right)}$ must be true if there is a solution to equation 6.

Reorganizing a little bit we come out with the inequality (10)

(10) $$k^{-1}x_m^{a_m}\sigma\left(\frac{k}{x_m^{a_m}}\right) < \frac{16}{15}$$

We can rewrite the left hand side of (10) as (11).

(11) $$\frac{\sigma(k)}{k}x_m^{a_m}\left(\frac{x_m - 1}{x_m^{a_m+1} - 1}\right) = \frac{\sigma(k)}{k}\left(\frac{x_m^{a_m+1} - x_m^{a_m}}{x_m^{a_m+1} - 1}\right)$$

The minimum of the x dependent part is 24/25 the minimum of sigma of k over k is 1.

Therefore odd perfect numbers if they exist must be such that.

(12) $$\frac{24}{25} < \frac{\sigma(k)}{k}\left(\frac{x_i - 1}{x_i^{a_i+1} - 1}\right) < \frac{16}{15}$$