Fourier Analysis

R.E.U. Report

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Preface

In science and particularly in mathematics, by and large, generalization of theorems is of great interest. This generalization can be delivered in two opposite directions: a) expanding the conclusions of a theorem into a bigger field, and b) minimizing or even in some senses changing the premises of a theorem.

In math 5210 I already had studied an introduction to Fourier analysis consisting of a chapter from the book, The Way of Analysis, by Strichartz, which more or less was dedicated to explanation the sufficient conditions for convergence of Fourier series. We chose the book "Introduction to Fourier Analysis and Wavelets" for my R.E.U. The writer, Mark Pinsky, spent the first chapter to examining the minimum or, necessary conditions of convergence, which I try to summarize hereunder. Alongside of theorems and explanations about Fourier series, some special techniques, used in the book were very instructive and were of great use in other fields of analysis in general. I summarize them as following:

1) Some techniques for estimating terms for example in Laplace's method, which is used in the proof of Sterling formula.
2) Using O (f (x)) and o (f (x)) to represent the limits. This representation sometimes is easier to work with, particularly is a criteria in computation of the rate of convergence.
3) There were many interesting applications of pointwise and uniform Holder and Lipschitz conditions in different forms and spaces.

4) There were many applications of the Lebesgue integral techniques and measure theory, for example, Fatou's lemma along with discrete measure is used to prove some interesting properties of Fourier series.

5) Some properties of the sine integral, **Sin (x),** which is the heart of many proofs of theorems about functions in L^1 (**T**).

Note: definite integrals in the following text are over the interval [-π, π], other wise it will be written.

A brief review

Corresponding to any function g (x) we can define a Fourier series as:

$$[1] \qquad \Sigma\hat{g}(n)\exp(int),\ n\epsilon\mathbb{Z} \text{ where}$$

$$\hat{g}(n)= \int g(\Phi)\exp(-in\Phi)d\Phi \quad -\pi\leq\Phi\leq\pi.$$

If g(x) is L^2(**T**) and we measure the distance by L^2 metric, then there are some interesting theorems for convergence in L^2 space as following:

1) Trigonometric system {exp(int)|n∊ ℤ} is orthonormal set in L^2(**T**),

2) The set of all trigonometric polynomials is dense in L^2(**T**), [Rudin, Real and Complex Analysis, page 89.]

3) Among all trigonometric polynomials of degree N, S (g, N) minimizes the L^2 distance to g, for any **2π** periodic continuous function.

We can conclude from 1), 2) and 3) that If g (x) be in L^2(**T**) then

$$[2] \qquad g(x)=\sum\hat{g}(n)\exp(inx)\ n\epsilon\ \mathbb{Z}. \quad \text{(converges in } L^2(\mathbf{T}))$$

Also we can show that if $\sum_{n\in\mathbb{Z}} n^2|\hat{g}(n)|^2<\infty$ then

$$g'(x)=\sum in\hat{g}(n)\exp(inx),\ n\epsilon\ \mathbb{Z}.$$

The following theorems are also true in L^2 (**T**).

Riesz-Fischer theorem: which asserts that L^2 (**T**) is isomorphic to $\bar{}^{2}(\mathbb{Z})$.

Parseval's theorem: which is the identity: $\sqrt{\int | g(\Phi)|^2 d\Phi} = \sum|\hat{g}(n)|^2$ where $n \in \mathbb{Z}$

and $\pi \le \Phi \le \pi$.

Bernstein's Theorem: if f satisfies L^2 Holder condition with $\alpha \geq ½$, then [1] is absolutely convergent.

Talking about convergence in L^∞ and pointwise convergence is more sophisticated subject and the following theorems are helpful:

Theorem: Let g be C^1 (**T**) then [1] is a uniformly convergent series and [2] holds. (Strichartz, The Way of Analysis, page 535)

Although this theorem enlightens the case for a big category of functions it has an internal deficiency because, as Strichrtz say, the premise, g be C^1 (**T**), is stronger than the conclusion. So it is natural to investigate some facts about the Fourier series of some bigger category of functions as C(**T**) or L^1 (**T**).

In C(**T**) or L^1 (**T**) the identity [2] doesn't hold in general. In 1926 Kolmogorov gave an example of L^1 function, g, for which the Fourier series [1] diverges everywhere in L^1 norm, and any set of measure Zero can be the set of points at which the Fourier series of a continuous function g diverges in the sup norm.

For functions in C(**T**) or L^1 (**T**) instead of investigating the convergence of their Fourier series we ask for another property called summability, which is weaker than being convergent. Cesaro's method of summability leads to Fejer's mean of first N partial sums of Fourier series. The following theorem can be helpful.

Fejer's theorem: the Fejer means of an L^1 (**T**) function converges uniformly almost everywhere to f in L^1 norm. More over if limits $f(\Theta\pm 0)$ exists then the Fejer means converges to $[f(\Theta-0)+f(\Theta+0)]/2$.

As we see Fejer means behave, as we desire even at point of discontinuity, while the Fourier series of functions even L^2 functions don't behave like this at the point of discontinuity. The behavior of Fourier series at point of discontinuity is called Gibbs phenomenon. The rest of this report is devoted to a few of problems that I solved.

1) Let f and g be L^1 (**T**). Show that for any bounded measurable function h we have:

$\iint h(x+y)\, f(x)\, g(y)\, dxdy = 2\pi\int h(z)(f*g)(z)\, dz$

Solution:

By definition: $2\pi\int h(z)(f*g)(z)\, dz = \int h(z) \int f(\Phi)\, g(z-\Phi)\, d\Phi dz$

$= \int\int h(z)\, f(\Phi)\, g(z-\Phi)\, d\Phi dz$

Now apply Fubini's theorem:

$=\int f(\Phi) \int h(z)\, g(z-\Phi)\, dz\, d\Phi$

Now put $z-\Phi=\zeta$:

$=\int f(\Phi) \int h(\zeta+\Phi)\, g(\zeta)\, d\zeta\, d\Phi$

2) Riemann- Lebesgue lemma: if $g(\Phi)$ is L^1 (**T**), $\hat{g}(n) \to 0$, as $|n|\to\infty$.

Solution:

The framework of the proof is in the book page 18. There in the book the conclusion is proved for simple functions and says that since simple functions are dense in L^1 the conclusion is correct in L^1. Here I prove the last statement:

Suppose g (N,.) is a simple function, which approaches g as $N\to\infty$, so:

$\hat{g}(n) = \int g(x) \exp(inx)\, dx = \int (g(x) - g(N,.)) \exp(inx)\, dx + \int g(N,.) \exp(inx)\, dx$

$|\int g(x) \exp(inx)\, dx| \leq \int |(g(x) - g(N,.))|\, dx + |\int g(N,.) \exp(inx)\, dx|$.

We can make the first term smaller than any given $\varepsilon/2$ by choosing N big enough and the second term will be less than $\varepsilon/2$ by choosing n big enough. So we get the conclusion.

3) Show that $\Omega_p(h+k) \leq \Omega_p(h) + \Omega_p(k)$ where Ω is the L^p modulus of continuity.

Solution:

$\Omega_p(h+k) = \sup \|f(x) - f(y)\|$, where sup is taken over all x and y such that $|x-y| \leq h+k$, so there exists z such that $\Omega_p(h+k) = \sup \|f(x) - f(y)\|$ where sup is taken over all x and y such that $|x-z| \leq h$ and $|y-z| \leq k$. So:

$\Omega_p(h+k) = \sup \|f(x) - f(z) + f(z) - f(y)\| \leq \sup\|f(x) - f(z)\| + \sup\|f(z) - f(y)\| = \Omega_p(h) + \Omega_p(k)$.

4) Suppose g is a function of B.V on T prove that for any f in L^1 (**T**):

$$\text{Lim} \int g(\Theta)\, s(N, f)(\Theta)\, d\Theta = \int g(\Theta)\, f(\Theta)\, d\Theta; \text{ as } N \to \infty$$

Solution:

$\int g(\Theta)\, s(N, f)(\Theta)\, d\Theta = \int g(\Theta)\, (D(N,.) * f)(\Theta)\, d\Theta$, since convolution is a self adjoint operator :

$= \int f(\Theta)\, (D(N,.) * g)(\Theta)\, d\Theta$. Now we know that $f(\Theta)\, (D(N,.) * g)(\Theta) \leq M f(\Theta) \in L^1$ (**T**), so applying the dominated convergence theorem we have:

$$\text{Lim} \int f(\Theta)\, (D(N,.) * g)(\Theta)\, d\Theta = \int \text{Lim}\, f(\Theta)\, (D(N,.) * g)(\Theta)\, d\Theta; \text{ as } N \to \infty$$

Since g is of B.V so $(D(N,.) * g)(\Theta) \to g(\Theta)$

So we have the conclusion.

5) Suppose that approximate identity $k(r, \Theta)$ is even and has the property that for each $\delta>0$ $\sup|k(r, \Theta)| \to 0$, where sup is taken over $|\Theta| \geq \delta$. Suppose that $\mathbf{\Phi} \in L^1(\mathbf{T})$, with $\lim[\mathbf{\Phi}(\Theta)+\mathbf{\Phi}(-\Theta)]=2L$ as $\Theta \to 0$, prove that

$$\text{Lim} \int k(r, \Theta)\, \mathbf{\Phi}(\Theta)\, d\Theta = L.$$ where limit is taken over r.

Solution:

$$\int k(r, \Theta)\, \mathbf{\Phi}(\Theta)\, d\Theta - L = \int_{|\Theta|>\delta} k(r, \Theta)[\mathbf{\Phi}(\Theta)-L]\, d\Theta + \int_{|\Theta|<\delta} k(r, \Theta)[\mathbf{\Phi}(-\Theta)-L]\, d\Theta + 0(1).$$

So the first integral in R.H.S is bounded by $\sup|k(r, \Theta)| * [\mathbf{\Phi}(\Theta)-L]$ which goes to zero, and the second integral can be arbitrary small by choosing δ small enough.