Bifurcation of the Spring Distribution in a Unit Long Homogeneous Beam

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Using the equation for the compliance of linearly elastic unit beam we have determined the points of bifurcation in a supporting spring distribution. Our beam is supported at two ends, loaded by some force distribution, $f(x)$, and supported by some spring distribution, $q(x)$. The deflection of the beam, the compliance, J, is governed by the force applied to the beam, the displacement vector, $w(x)$, Young's Modules, and the spring distribution. For simplicity Young's Modules has been set to one.

$$J = \int_0^1 f \cdot w - \frac{E}{2}(w'')^2 - \frac{q}{2}w^2 dx$$

To formulate the problem we begin by varying the functional, J, with respect to the displacement vector. Here we must generalize the Euler-Lagrange equation to include variation with respect to second derivatives.

$$\frac{\partial L}{\partial w} - \frac{d}{dx}\left(\frac{\partial L}{\partial w'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial L}{\partial w''}\right) = 0$$

We find the equation that the optimal $w(x)$ must satisfy. This equation coupled with boundary conditions provides $w(x)$. Note that in the variation that I have conducted to arrive at the following equation, I have included a constraint on $f(x)$ in the functional, J.

$$\left(Ew''\right)'' - qw = f$$

Key to the formulation of this problem is the principal compliance, Λ. The principal compliance is a quantity that we find by maximizing the functional with respect to $f(x)$ the force distribution. It is important to note that the force is constrained, and thus, we maximize the compliance for any force in a set F. In order to optimize the beam we minimize the principal compliance; carried out by varying the new functional, Λ, with respect to the constrained spring distribution. Consequently the structure is now optimized for any load in the set of forces, F.

$$F = \{f : \int_0^1 f(x)dx = 1\}$$

$$\max_{f \in F} J = \Lambda$$

In order to analyze the problem we must know $f(x)$. Varying the functional classically yields a trivial result, and thus we have to allow $f(x)$ to be a sum of delta functions. If we allow $f(x)$ to be a sum of delta functions, $\sum_i \delta(x-x_i)$, the compliance can be maximized if the first derivative of the displacement vector vanishes at x_i. However, one delta function is optimal[1] $f_D = \delta(x-x_i)$, where f_D is the maximizing $f(x) \in F$. Further through the same reasoning, it can be shown that the optimal spring distribution is also a sum of delta functions.

$$q = \sum_i \alpha_i \delta(x - X_i)$$

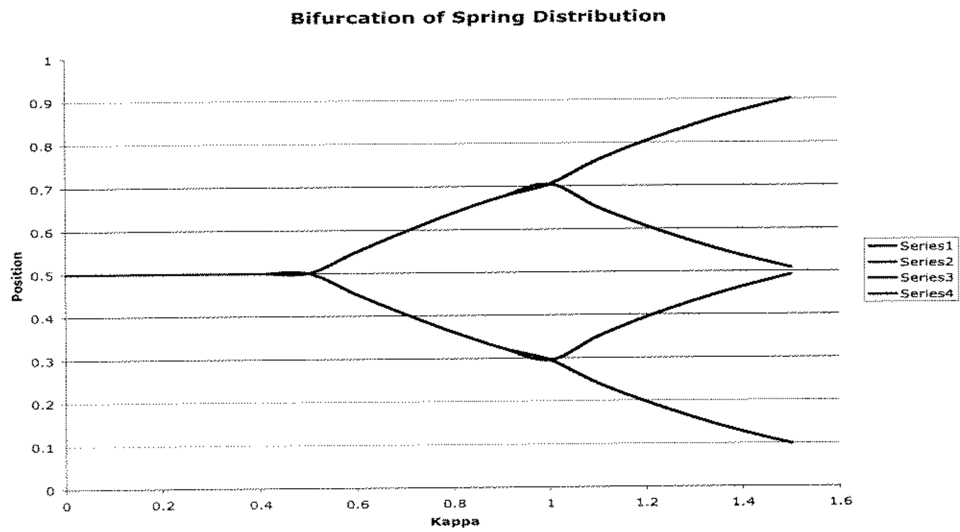
The constraints are as follows

$$\sum_i \alpha_i = \kappa .$$

Knowing $f(x)$ and $q(x)$ and using the elastic equation, we see that $w(x)$ is a piecewise continuous function of α_n and X_n. In the equation for $w(x)$ below, m+1 (m=0,1,2..) refers to the segment of the beam. The segments are defined by the positions of the springs. In this equation total is the total number of forces on the beam, supporting and loading, and X_n is the position of the forces.

$$w_{m+1} = \frac{1}{6}\left(\sum_{n=1}^{total} q_n - \sum_{n=1}^{total} X_n q_n - \sum_{n=1}^{m} q_n\right)x^3 + \frac{1}{2}\sum_{n=1}^{m} X_n q_n x^2 + \left(-\frac{1}{3}\sum_{n=1}^{total} X_n q_n - \frac{1}{6}\sum_{n=1}^{total} X_n^3 q_n + \frac{1}{2}\sum_{n=1}^{total} X_n^2 q_n - \frac{1}{2}\sum_{n=1}^{m} X_n^2 q_n\right)x + \frac{1}{6}\sum_{n=1}^{m} X_n^3 q_n$$

This is where we encounter bifurcation. Given a fixed constraint on $f(x)$ the number and position of the delta functions in $q(x)$ is a function of the constraint, κ. Below is a graph showing spring position as a function of κ. It is important to note here that after the first bifurcation this is a gross simplification of the problem. I have only allowed for the spring's positions to change symmetrically, and have restricted their individual weights to be equal. This example is restricted to four springs, with symmetric position and equal values for α_n, which is reasonable for $f = \delta(x - 1/2)$. The first bifurcation occurs when α_n equals 0.25 (two springs), and the second when α_n equals 0.25 (four springs).



I studied cases in which involved three springs as well; bifurcation occurred when κ was equal to the magnitude of the force. However, continuing past these two simple examples has proved computationally rigorous. To analyze the next bifurcation one must necessarily double the number of unknowns. Due to the sheer size of the principal compliance, the server has not been able to successfully compute the bifurcation for several springs.