REU PROJECT REPORT

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I first happened upon the subject of frames when investigating the subject of wavelets in a signals processing text. We define a frame as a (redundant) set of vectors $\{\phi_n\}_{n\in I}$ in a Hilbert space H if there are numbers $A, B \geq 0$ such that for all $\phi \in H$,

$$A\|\phi\|^2 \leq \sum_{n\in I} |\langle \phi, \phi_n \rangle|^2 \leq B\|\phi\|^2.$$

Frames are important in signals processing because signals decomposed into coordinates, transmitted and then reconstructed back onto the original frame are usually quite robust. However, the subject also draws from some of the deeper and more interesting areas of mathematics, which makes it worth examining in and of itself. The remainder of this report will be a survey of results and topics studied throughout the summer.

The first introduction to the subject I considered was from Mallat's[1] extremely user-friendly introduction to the topic, presented from a signal processing point of view. His chapter about frames discusses them in the context of trigonometric polynomials, and completing the exercises helped develop some intuition about the subject. Ole Christiansen's[2] book is another good reference on the topic, as his development is slow and rigorous, and the exercises he includes are difficult and rewarding. Of the different tributaries that feed into the study of frames, those covered under the umbrella of functional analysis came up much more than anything else. For these, I found Serge Lang's book on the subject to be a good introduction, but by far the most relevant and useful I have encountered is Akhiezer and Glazman's Theory of Linear Operators in Hilbert Space[3]. The subject of frames is treated almost universally in the context of Hilbert spaces, so it makes sense to limit the study to these. Frames are often generated by and defined through operators, so this text was particularly useful in that regard. Two specific operators are of general importance in consideration of frames, the translation and modulation operators, are defined as follows:

$$\text{Translation by } a \in \mathbb{R},\ T_a\colon\ L^2(\mathbb{R}) \to L^2(\mathbb{R}), (T_a f)(x) = f(x-a),$$

and

$$\text{Modulation by } b \in \mathbb{R},\ E_b\colon\ L^2(\mathbb{R}) \to L^2(\mathbb{R}), (M_b f)(x) = e^{2\pi ibx} f(x).$$

Alkhiezer's book is also a good introduction to some important definitions and properties. A subspace $H_1 \subset H$ is called an *invariant subspace* of a given operator T if every element $f \in D_T$, where $D_T \subset H_1$ is mapped by the operator T into an element also belonging to

H_1, i.e., $f \in D_T \cup H_1 \Rightarrow Tf \in H_1$. It is more or less a generalization of the concept of eigenvalues and eigenmanifolds from linear algebra. Techniques of analyzing frames based on these ideas are often quite fruitful, predominantly because one cleverly chooses a subspace that is invariant for a particular operator. For Gabor frames, shift-invariant and modulation-invariant subspaces are essential for reducing the complexity of a system owing to the fact that any properties one discovers in these subspaces are true in general for the operators and space in its entirety. We attempted to show that there are invariant subspaces for both the Translation and Modulation operators by finding them directly, and investigated some articles on the topic that were specific to Gabor frames.

In terms of the modulation and translation operators defined above, a Gabor frame is a frame for $L^2(\mathbb{R})$of the form $E_{mb}T_{na,(m,n)\in\mathbb{Z}}$, where $a, b > 0$ and $g \in L^2(\mathbb{R})$ is a fixed function. The function g is often referred to as the window function or generator, and these new frames are often called Weyl-Heisenberg frames, because they can be characterized as a representation of the Heisenberg matrix group.

A good example of why it is important to consider Gabor frames, as opposed to a orthonormal basis, is the Balian-Low theorem. The Gabor orthonormal basis is defined with the indicator function as $E_mT_n\chi_{[0,1]}(x)_{m,n\in\mathbb{Z}}$. If the indicator function is replaced by a different function that is continuous and has more desirable decay properties, such as a Gaussian, then this function cannot be well localized in both time and frequency. That is, at least one of Fourier transforms of either must diverge. Formally, the theorem states that if E_mT_ng is a Riesz basis for $L^2(\mathbb{R})$, where m and n are integers, then $(\int_{-\infty}^{\infty}|xg(x)|^2dx)(\int_{-\infty}^{\infty}|\gamma g(\gamma)|^2dx) = \infty$.

Christiansen's book offers the following conjecture: Given any finite collection of distinct points $(\mu_k, \lambda_k)_{k\in\mathbb{I}}$ in $\mathbb{R}^2$ and a function $g \neq 0$, the Gabor system $e^{2\pi i\lambda_k x}g(x - mu_k)_{k\in\mathbb{I}}$ is linearly independent. In light of the operators mentioned above, finite subsets of a regular Gabor frame must be linearly independent.

I invested some time trying to understand this problem and why it is so difficult to solve. My initial investigation was fairly rudimentary; Using matlab, I set up a few different window functions, created a Gabor frame with 2 or 3 elements, and attempted to solve the equation $c_1g_1 + c_2g_2 + \cdots + c_ng_n = 0$ numerically for Gabor functions $g_n(x)$ and for values of $x \in \mathbb{R}$. In the cases I considered, the functions were all likely linearly independent, because the program couldn't find any value that satisfied the relation. It turns out that, in the special case where $(\mu_k, \lambda_n) = (na, mb)_{n=1,m=1}^{N,M}$, or the indexes of λ and μ are a lattice of integers, the finite Gabor frame is linearly independent. The proof of this relies on the properties of the operators themselves, and their relationship with Von-Neumann algebras. At this time there is no simple, well-defined perturbation method to analyze the system when the indexes are not integers, which is why the problem has been so difficult to for non-integer values. Nevertheless, there is no counterexample, and most believe that the conjecture is indeed true.

The final two topics included in my program of study this summer were Deguang Han and David Larson's article[4] which contains some interesting theorems on the subject of Gabor frames and wandering vectors, and the Grassmannian frames, described in Strohmer et al.[5]. Given some conditions on the frame coefficients, it is possible that every wandering vector in a given frame is complete. The subject also has some connections to algebras, which I explored briefly. Grassmannian frames are those that minimize the maximum inner product between any two vectors in a given frame. These are highly desirable for signal processing and communications, but very little is actually known about them. Most of the theorems I investigated borrowed heavily from sphere packing and other areas of mathematics that were developed independently of frame theory.

The subject of frame theory has some very rich and useful results, and merits further investigation. Finally, I would like to offer my sincerest thanks to my mentor, Dr. Nathan Smale, and the University of Utah Department of Mathematics , for providing me with this extraordinary and fun opportunity.

References

[1] Mallat, Stephane. A Wavelet Tour of Signal Processing. Elsevier, 1999. pgs. 125-160.

[2] Christensen, Ole. An Introduction to Frames and Riesz Bases. Birkhauser, 2003.

[3] N.I. Akhiezer and I.M. Glazman. Theory of Linear Operators in Hilbert Space. Dover, 1993.

[4] Deguang Han and David R. Larson. Frames, Bases and Group Representations. Mem. Amer. math. Soc. 147 no. 697, 2000.

[5] Thomas Strohmer and Robert Heath, Grassmannian Frames with Applications to Coding and Communication. Appl. Comp. Harm. Anal, vol.14(3), 2003. pgs. 257-275.