CURVE COMPLEXES AND FINITE INDEX SUBGROUPS OF MAPPING CLASS GROUPS

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ABSTRACT. Let $\mathrm{Mod}(S)$ be the extended mapping class group of a surface S. For S the twice-punctured torus, we show that there exists an isomorphism of finite index subgroups of $\mathrm{Mod}(S)$ which is not the restriction of any inner automorphism. For S a torus with at least three punctures, we show that every injection of a finite index subgroup of $\mathrm{Mod}(S)$ into $\mathrm{Mod}(S)$ is the restriction of an inner automorphism of $\mathrm{Mod}(S)$; this completes a program begun by Irmak. We also establish the co-Hopf property for finite index subgroups of $\mathrm{Mod}(S)$.

1. INTRODUCTION

Let $S = S_{g,n}$ be a *surface*, by which we always mean a connected, orientable surface of genus g with n punctures. The *extended mapping class group* of S is:

$$\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^{\pm}(S))$$

In this note we complete the answer to the following question:

Question. *Given a surface S, is it true that every injection of a finite index subgroup of $\mathrm{Mod}(S)$ into $\mathrm{Mod}(S)$ is the restriction of an inner automorphism of $\mathrm{Mod}(S)$?*

The study of maps of finite index subgroups of $\mathrm{Mod}(S)$ into $\mathrm{Mod}(S)$ was begun by Ivanov, although this specific question was first investigated by Irmak, who showed that the answer is no for $S_{2,0}$ (see below) and yes for all other $S_{g,n}$ with $g \geq 2$ [17] [16] [15]. Bell–Margalit proved

that the answer is yes for $S_{0,n}$ with $n \geq 5$, giving applications to the study of Artin groups [3].

Korkmaz showed that the answer is no for $S_{0,4}$, $S_{1,1}$, and $S_{1,0}$, as the mapping class groups for these surfaces contain free subgroups with finite index [23]. Also, the answer is no for $S_{0,2}$ and $S_{0,3}$ and yes for $S_{0,0}$ and $S_{0,1}$; this is not hard to check since the mapping class groups for these surfaces are finite.

In this paper, we treat all remaining cases. For $S_{1,2}$, we show that the answer is no:

Main Theorem 1. *There exists an isomorphism between finite index subgroups of* $\mathrm{Mod}(S_{1,2})$ *which is not the restriction of an inner automorphism of* $\mathrm{Mod}(S_{1,2})$.

We show that the answer is yes in the other cases:

Main Theorem 2. *For $n \geq 3$, any injection of a finite index subgroup of* $\mathrm{Mod}(S_{1,n})$ *into* $\mathrm{Mod}(S_{1,n})$ *is the restriction of an inner automorphism of* $\mathrm{Mod}(S_{1,n})$.

We thus have the following theorem, which combines our results with those of Korkmaz, Irmak, and Bell–Margalit.

Theorem 3. *If S is a surface which is not $S_{0,2}$, $S_{0,3}$, $S_{0,4}$, $S_{1,0}$, $S_{1,1}$, $S_{1,2}$, or $S_{2,0}$, then every injection of a finite index subgroup of* $\mathrm{Mod}(S)$ *into* $\mathrm{Mod}(S)$ *is the restriction of an inner automorphism of* $\mathrm{Mod}(S)$. *If S is one of these exceptional surfaces, then there is an isomorphism of finite index subgroups of* $\mathrm{Mod}(S)$ *which is not the restriction of an inner automorphism of* $\mathrm{Mod}(S)$.

In the case of $S_{2,0}$, if Γ is a finite index subgroup of $\mathrm{Mod}(S_{2,0})$ and $\rho : \Gamma \rightarrow \mathrm{Mod}(S_{2,0})$ is an injection, then ρ is given by $\rho(g) = f g f^{-1} \iota^{\sigma(g)}$, where $f \in \mathrm{Mod}(S_{2,0})$, ι is the hyperelliptic involution of $S_{2,0}$, and $\sigma : \Gamma \rightarrow \mathbb{Z}_2$ is a homomorphism; note $\langle \iota \rangle = Z(\mathrm{Mod}(S_{2,0})) \cong \mathbb{Z}_2$. It is not hard to construct examples of injections where the associated homomorphism σ is nontrivial; any such injection is not the restriction of an inner automorphism of $\mathrm{Mod}(S_{2,0})$. In this case, if Γ' is the kernel of σ, then $\rho|_{\Gamma'}$ is the restriction of an inner automorphism and the index of Γ' in Γ is 2.

Complex of curves and Ivanov's theorem. Let $\mathcal{C}(S)$ denote the *complex of curves* for S, which is the abstract simplicial flag complex

with a vertex for each isotopy class of simple closed curves in S and an edge between vertices with disjoint representatives; this complex was defined by Harvey [13]. In his seminal work, Ivanov proved that every isomorphism between finite index subgroups of $\mathrm{Mod}(S)$ is the restriction of an inner automorphism; the main step was to show that every automorphism of $\mathcal{C}(S)$ is induced by an element of $\mathrm{Mod}(S)$ [19].

In light of Ivanov's theorem, Theorem 3 can be thought of as saying that every injection of a finite index subgroup of $\mathrm{Mod}(S)$ has finite index image in $\mathrm{Mod}(S)$. However, showing that the image of such an injection has finite index does not seem to be easier than showing directly that the injection is the restriction of an inner automorphism.

Superinjective maps. To attack the particular question at hand, Irmak introduced the notion of a *superinjective* map of $\mathcal{C}(S)$, which is a simplicial map of $\mathcal{C}(S)$ to itself preserving disjointness and nondisjointness of the isotopy classes of curves corresponding to the vertices of $\mathcal{C}(S)$ (note any simplicial map of $\mathcal{C}(S)$ preserves disjointness).

It is clear that elements of $\mathrm{Mod}(S)$ give rise to superinjective maps of $\mathcal{C}(S)$. Following Irmak, Main Theorem 2 is proven by showing that all superinjective maps arise in this way:

Theorem 4. *For $n \geq 3$, every superinjective map of $\mathcal{C}(S_{1,n})$ is induced by an element of $\mathrm{Mod}(S_{1,n})$.*

For this theorem, the main difficulty lies in distinguishing nonseparating curves from curves bounding twice-punctured disks. To overcome this obstacle, we introduce an essential new tool, the *adjacency graph* of a pants decomposition of a surface (see Section 5). This notion can also be used to provide a new approach to Theorem 4 for the cases not covered in this paper.

After the completion of this work, we learned that the idea of the adjacency graph was independently and simultaneously discovered by Shackleton. He used it to show (in most cases) that any simplicial embedding of a curve complex into a curve complex of equal or lesser dimension is necessarily an automorphism [27]. This reproves Theorem 3 in many cases, and also has the consequence that, for these mapping class groups, there are no injections of a finite index subgroup into a mapping class group of equal or lesser complexity.

The deduction of Main Theorem 2 from Theorem 4 is now a standard argument for which we refer the reader to Irmak's paper [17]. The idea is that an injection of a finite index subgroup of $\text{Mod}(S)$ into $\text{Mod}(S)$ must take powers of *Dehn twists* to powers of Dehn twists. These are exactly the elements of $\text{Mod}(S)$ which are each supported on the regular neighborhood of a simple closed curve in S (see e.g. [4]). Powers of Dehn twists commute if and only if the corresponding curves are disjoint, so the injection gives a superinjective map of $\mathcal{C}(S)$.

Conventions. We refer to essential simple closed curves in a surface as *curves*. When there is no confusion we will blur the distinction between a curve and its isotopy class. Two curves a and b are said to be *disjoint* when their geometric intersection number, $\text{i}(a, b)$, is 0. An *arc* is the isotopy class of the image of a proper, essential embedding of $\mathbb{R}$ in a surface; here, *essential* means not homotopic to a puncture.

2. The co-Hopf property

A group G is *co-Hopfian* if every injective endomorphism of G is an automorphism of G, and it is *Hopfian* if every surjective endomorphism is an automorphism. In general, the former property seems to be more rare and harder to prove than the latter.

Grossman proved that mapping class groups are residually finite [11] (see also [18]). Since finitely generated residually finite groups are Hopfian, we have this property for $\text{Mod}(S)$.

It follows formally from Theorem 3 that $\text{Mod}(S)$ is co-Hopfian for most surfaces S. This was first proven by Ivanov–McCarthy [20]. What is more, it is straightforward to deduce from Theorem 3 the following:

Theorem 5. *If S is not $S_{0,4}$, $S_{1,0}$, or $S_{1,1}$, then every finite index subgroup of $\mathrm{Mod}(S)$ is co-Hopfian.*

Note that the theorem is not true for S either $S_{0,4}$, $S_{1,0}$, or $S_{1,1}$, by the fact that $\mathrm{Mod}(S)$ contains a free group with finite index in these cases. Since they are not treated by Theorem 3, the surfaces $S_{2,0}$ and $S_{1,2}$ require special attention. The case of $S_{2,0}$ is handled by the statement immediately following Theorem 3 and an elementary group theory argument. We relegate the proof for $S_{1,2}$ to Section 4.3.

Related results. Farb–Ivanov showed that the *Torelli group*, the subgroup of $\mathrm{Mod}(S)$ acting trivially on the homology of S, is co-Hopfian [10]. Brendle–Margalit proved that the so-called *Johnson kernel* and all of its finite index subgroups are co-Hopfian [7]; this result and the previous were proven for closed surfaces of genus at least 4. Bell–Margalit established the co-Hopf property for $B_n/Z(B_n)$ with $n \geq 4$, where B_n is the *braid group on n strands* [2]. Recently, Farb–Handel showed that $\mathrm{Out}(F_n)$ and all of its finite index subgroups are co-Hopfian for $n \geq 4$ [9].

3. ABSTRACT COMMENSURATORS

To better understand Main Theorem 1, it will be helpful to recast Ivanov's theorem by putting a group structure on the set of isomorphisms of finite index subgroups of $\mathrm{Mod}(S)$.

The *abstract commensurator* $\mathrm{Comm}(G)$ of a group G is the group of equivalence classes of isomorphisms of finite index subgroups of G. Two isomorphisms are said to be equivalent if they agree on a finite index subgroup of G. The composition of two isomorphisms $\psi : \Gamma \to \Lambda$ and $\psi' : \Gamma' \to \Lambda'$ is a map defined on the finite index subgroup $\psi^{-1}(\Lambda \cap \Gamma')$. A simple example is $\mathrm{Comm}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Q})$.

In this language, Ivanov's result is:

Theorem 6. *Let S be any surface other than $S_{1,n}$ with $n \leq 2$ or $S_{0,n}$ with $n \leq 4$. The natural homomorphism*

$$\mathrm{Mod}(S) \to \mathrm{Comm}(\mathrm{Mod}(S))$$

is surjective; its kernel is $Z(\mathrm{Mod}(S)) \cong \mathbb{Z}_2$ for $S = S_{2,0}$ and is trivial otherwise.

This theorem was proven by Ivanov for surfaces of genus at least 2 [19]. Korkmaz then proved it for surfaces of genus 0 and 1 [22]. Theorem 6 can also be deduced from work of Luo, who gave a new proof that the natural map $\mathrm{Mod}(S) \to \mathrm{Aut}(\mathcal{C}(S))$ is surjective [25].

We remark that Theorem 6 is a corollary of Theorem 3. To see why Theorem 3 is a priori more difficult, note that, by the same logic used to deduce Main Theorem 2 from Theorem 4 (see Section 1), an element of $\mathrm{Comm}(\mathrm{Mod}(S))$ gives rise to an *automorphism* of $\mathcal{C}(S)$. For the non-exceptional surfaces, the topological type of a curve is determined by the homotopy type of its link in $\mathcal{C}(S)$ (this is a theorem of Harer [12]). It follows immediately that an element of $\mathrm{Aut}(\mathcal{C}(S))$ preserves the topological type of a curve. For a superinjective map of $\mathcal{C}(S)$, this is a difficult step. Still, our argument follows the general outline of Ivanov's proof.

Related results. By work of Farb–Ivanov and Brendle–Margalit, the abstract commensurators of the Torelli group and the Johnson kernel are both isomorphic to $\mathrm{Mod}(S)$ for closed surfaces of genus at least 4 [10] [7]. Charney–Crisp used Theorem 6 to show that the abstract commensurators of several affine and finite type Artin groups, modulo their centers, are isomorphic to $\mathrm{Mod}(S_{0,n})$ [8]. Leininger–Margalit proved that $\mathrm{Comm}(B_n) \cong \mathrm{Mod}(S_{0,n+1}) \ltimes (\mathbb{Q}^\times \ltimes \mathbb{Q}^\infty)$ when $n \geq 4$ [24], and Farb–Handel showed $\mathrm{Comm}(\mathrm{Out}(F_n)) \cong \mathrm{Out}(F_n)$ for $n \geq 4$ [9].

4. TWICE-PUNCTURED TORUS

Luo showed that $S_{1,2}$ has the exceptional property that there are automorphisms of $\mathcal{C}(S_{1,2})$ which are not induced by $\mathrm{Mod}(S_{1,2})$ [25]. We will translate this fact into the group theoretic statement of Main Theorem 1. We then establish Theorem 5 for $S_{1,2}$, and also describe all injections of finite index subgroups of $\mathrm{Mod}(S_{1,2})$ into $\mathrm{Mod}(S_{1,2})$.

4.1. **Nongeometric curve complex automorphisms.** We start by giving the argument of Luo. The first step is to construct an isomorphism between $\mathcal{C}(S_{0,5})$ and $\mathcal{C}(S_{1,2})$. Let ι be the hyperelliptic involution of $S_{1,2}$. The quotient $S_{1,2}/\langle \iota \rangle$ is a sphere with 1 puncture and 4 cone points of order 2. Identify $S_{0,5}$ with the complement of a regular neighborhood of these cone points. Given any curve in $S_{0,5}$, we can lift it via ι to a curve in $S_{1,2}$. Moreover, this map preserves disjointness. Also, this map is surjective: Birman and Viro showed that ι fixes every curve

in $S_{1,2}$ (being central, it commutes with each Dehn twist), and so there is an inverse map [4] [28].

In $S_{0,5}$, all curves are topologically equivalent, since they each have three punctures on one side and two on the other. Thus, $\mathrm{Aut}(\mathcal{C}(S_{0,5}))$ acts transitively on the vertices of $\mathcal{C}(S_{0,5})$. Since $\mathcal{C}(S_{1,2})$ is isomorphic to $\mathcal{C}(S_{0,5})$, it follows that there are elements of $\mathrm{Aut}(\mathcal{C}(S_{1,2}))$ which interchange separating and nonseparating curves. Such automorphisms clearly cannot be induced by $\mathrm{Mod}(S_{1,2})$.

To see this more concretely, note that the curves in $S_{0,5}$ corresponding to separating curves in $S_{1,2}$ are exactly the curves which contain the *special puncture* (the one coming from the punctures of $S_{1,2}$) on their twice-punctured sides.

4.2. **Nongeometric commensurators.** We now show that the elements of $\mathrm{Aut}(\mathcal{C}(S_{1,2}))$ which are not induced by $\mathrm{Mod}(S_{1,2})$ give rise to elements of $\mathrm{Comm}(\mathrm{Mod}(S_{1,2}))$ which are not induced by $\mathrm{Mod}(S_{1,2})$.

Let $\mathrm{PMod}(S_{1,2})$ denote the subgroup of $\mathrm{Mod}(S_{1,2})$ consisting of elements which fix each puncture. By work of Birman–Hilden, ι induces an isomorphism $\iota_\star$ of $\mathrm{PMod}(S_{1,2})$ with the subgroup of $\mathrm{Mod}(S_{0,5})$ consisting of elements which fix the special puncture [5]. This index 5 subgroup of $\mathrm{Mod}(S_{0,5})$ corresponds exactly to the subgroup of $\mathrm{Aut}(\mathcal{C}(S_{0,5}))$ consisting of elements which lift to automorphisms of $\mathcal{C}(S_{1,2})$ induced by $\mathrm{Mod}(S_{1,2})$.

Combining Theorem 6 with the fact that $\mathrm{Mod}(S_{1,2})$ and $\mathrm{Mod}(S_{0,5})$ have isomorphic finite index subgroups, it follows that:

Proposition 7. $\mathrm{Comm}(\mathrm{Mod}(S_{1,2})) \cong \mathrm{Comm}(\mathrm{Mod}(S_{0,5})) \cong \mathrm{Mod}(S_{0,5})$.

Now, consider the composition

$$\mathrm{Mod}(S_{1,2}) \to \mathrm{Comm}(\mathrm{Mod}(S_{1,2})) \to \mathrm{Comm}(\mathrm{Mod}(S_{0,5})) \to \mathrm{Mod}(S_{0,5})$$

where $g \in \mathrm{Mod}(S_{1,2})$ maps to conjugation by g, which then maps to conjugation by $\iota_\star(g)$, and finally to the element $\iota_\star(g)$ of $\mathrm{Mod}(S_{0,5})$. We have already noted that $\iota_\star(\mathrm{Mod}(S_{1,2}))$ has index 5 in $\mathrm{Mod}(S_{0,5})$. Since the kernel of the first map is $Z(\mathrm{Mod}(S_{1,2})) = \langle \iota \rangle \cong \mathbb{Z}_2$ and the last two maps are restrictions of the isomorphisms of Proposition 7, this yields the following, which implies Main Theorem 1.

Proposition 8. $[\mathrm{Comm}(\mathrm{Mod}(S_{1,2})) : \mathrm{Mod}(S_{1,2})/\mathbb{Z}_2] = 5$.

We now explain why the elements of $\mathrm{Comm}(\mathrm{Mod}(S_{1,2}))$ which do not come from $\mathrm{Mod}(S_{1,2})$ are not geometric in any sense.

If $\mathrm{PMod}(S_{0,5})$ is defined similarly to $\mathrm{PMod}(S_{1,2})$, then $\mathrm{PMod}(S_{0,5})$ is normal in $\mathrm{Mod}(S_{0,5})$ and is isomorphic (via $\iota_\star$) to a finite index subgroup Γ of $\mathrm{Mod}(S_{1,2})$. Now, we can write the isomorphism

$$\Upsilon : \mathrm{Mod}(S_{0,5}) \to \mathrm{Comm}(\mathrm{Mod}(S_{1,2}))$$

as:

$$g \mapsto [\Gamma \to \iota_\star^{-1}(g\iota_\star(\Gamma)g^{-1})]$$

Denote by T_c the Dehn twist about a curve c, and let g be an arbitrary element of $\mathrm{Mod}(S)$. From the general formula

$$(1) \qquad\qquad\qquad gT_c^k g^{-1} = T_{g(c)}^{\pm k}$$

it follows that any representative of $\Upsilon(g)$ takes powers of T_c to powers of $T_{g_\star(c)}$, where $g_\star$ is the induced element of $\mathrm{Aut}(\mathcal{C}^0(S_{1,2}))$ ($\mathcal{C}^0(S_{1,2})$ is the 0-skeleton). Since any element of $\mathrm{Mod}(S_{0,5}) - \iota_\star(\mathrm{Mod}(S_{1,2}))$ gives an element of $\mathrm{Aut}(\mathcal{C}^0(S_{1,2}))$ interchanging separating and nonseparating curves, it follows that the images under Υ of these elements take powers of Dehn twists about separating curves to powers of Dehn twists about nonseparating curves (and vice versa). By contrast, equation (1) shows that an inner automorphism preserves the topological types of curves corresponding to powers of Dehn twists.

Remark. The elements of $\mathrm{Comm}(\mathrm{Mod}(S_{1,2}))$ which do not arise from inner automorphisms of $\mathrm{Mod}(S_{1,2})$ actually do not arise from any elements of $\mathrm{Aut}(\mathrm{Mod}(S_{1,2}))$. This follows from the fact that, in $\mathrm{Mod}(S_{1,2})$, conjugate Dehn twists can commute only if they are twists about nonseparating curves. Thus, $\mathrm{Aut}(\mathrm{Mod}(S_{1,2})) \cong \mathrm{Mod}(S_{1,2})/\langle \iota \rangle$.

4.3. **Proof of Theorem 5.** Despite the fact that there are nongeometric elements of $\mathrm{Comm}(\mathrm{Mod}(S_{1,2}))$, we can still show that finite index subgroups of $\mathrm{Mod}(S_{1,2})$ are co-Hopfian. The first step is:

Proposition 9. $\mathrm{Mod}(S_{1,2}) \cong \mathrm{PMod}(S_{1,2}) \times \mathbb{Z}_2$.

This proposition follows from the fact that the short exact sequence

$$1 \to \langle \iota \rangle \to \mathrm{Mod}(S_{1,2}) \to \mathrm{Mod}(S_{1,2})/\langle \iota \rangle \to 1$$

has a splitting $\mathrm{Mod}(S_{1,2})/\langle \iota \rangle \to \mathrm{Mod}(S_{1,2})$ defined by sending a coset to its unique representative in $\mathrm{PMod}(S_{1,2})$.

Since $\mathrm{PMod}(S_{1,2})$ is isomorphic to a finite index subgroup of $\mathrm{Mod}(S_{0,5})$, every finite index subgroup of $\mathrm{PMod}(S_{1,2})$ is co-Hopfian, as Theorem 5 follows from Theorem 3 in the case of $S_{0,5}$. Using Proposition 9 and the fact that finite index subgroups of $\mathrm{Mod}(S_{0,5})$ have trivial center, an elementary group theory argument shows that every finite index subgroup of $\mathrm{Mod}(S_{1,2})$ is co-Hopfian.

4.4. **Injections of finite index subgroups.**

If K is any finite index subgroup of $\mathrm{PMod}(S_{1,2}) \times \mathbb{Z}_2 \cong \mathrm{Mod}(S_{1,2})$, then K is isomorphic to $G \times H$, where G is a finite index subgroup of $\mathrm{PMod}(S_{1,2})$ and $H < \mathbb{Z}_2$. We fix inclusions:

$$G \hookrightarrow \mathrm{PMod}(S_{1,2}) \hookrightarrow \mathrm{Mod}(S_{0,5})$$

By an argument similar to that needed for Section 4.3, we can describe all injections of finite index subgroups of $\mathrm{Mod}(S_{1,2})$ into $\mathrm{Mod}(S_{1,2})$:

Proposition 10. *Let ρ be an injection $K \to \mathrm{Mod}(S_{1,2})$, thought of as $G \times H \to \mathrm{PMod}(S_{1,2}) \times \mathbb{Z}_2$. Then ρ is given by*

$$(g, x) \mapsto (\Omega(g), \sigma(g) \cdot x)$$

where $\Omega \in \mathrm{Aut}(\mathrm{Mod}(S_{0,5})) \cong \mathrm{Mod}(S_{0,5})$ and $\sigma : G \to \mathbb{Z}_2$ is a homomorphism.

When, in the conclusion of the proposition, Ω is trivial, we remark that ρ is a *transvection* in a sense slightly generalizing that of Charney–Crisp [8] (see also [3] [24]).

5. Proof of Theorem 4

In this section, we fix a superinjective map $\phi : \mathcal{C}(S_{1,n}) \to \mathcal{C}(S_{1,n})$, where $n \geq 3$. We will prove Theorem 4 in this case; that is, we will show that ϕ is induced by an element of $\mathrm{Mod}(S_{1,n})$.

We start by showing that ϕ preserves certain topological properties of (and relationships between) curves.

Pants decompositions. It is not hard to see that any superinjective map is injective (for any two curves, consider a curve which is disjoint from one but not the other). It follows that ϕ preserves the set of *pants decompositions* of $S_{1,n}$, i.e. the maximal simplices of $\mathcal{C}(S_{1,n})$.

Adjacency graphs. Given a pants decomposition $\mathcal{P}$ of $S_{1,n}$, we say that two curves of $\mathcal{P}$ are *adjacent with respect to* $\mathcal{P}$ if there exists a component of $S_{1,n} - \mathcal{P}$ containing both curves in its closure; this definition is essential in the work of Irmak [17] [16]. We assign an *adjacency graph* $\mathcal{G}(\mathcal{P})$ to $\mathcal{P}$ consisting of a vertex for each curve in $\mathcal{P}$ and edges corresponding to adjacency.

The following result provides our main tool, and the proof we give works for all surfaces.

Lemma 11. *For any pants decomposition $\mathcal{P}$, ϕ induces an isomorphism of the graphs $\mathcal{G}(\mathcal{P})$ and $\mathcal{G}(\phi(\mathcal{P}))$.*

Proof. It suffices to show ϕ preserves adjacency and nonadjacency with respect to $\mathcal{P}$.

Two curves are adjacent with respect to $\mathcal{P}$ if and only if there exists a curve which intersects both and is disjoint from the other curves of $\mathcal{P}$. Since superinjective maps preserve disjointness and nondisjointness, this characterization implies that ϕ preserves adjacency.

Let $\mathcal{P} = \{a_1, b_1, c_1, \ldots, c_{n-2}\}$. If a_1 and b_1 are not adjacent with respect to $\mathcal{P}$, then there exist curves a_2 and b_2 so that $\{a_i, b_j, c_1, \cdots, c_{n-2}\}$ is a pants decomposition for any $i, j \in \{1, 2\}$. Since curves in a pants decomposition are disjoint, it follows that a_1, b_1, a_2, and b_2 (in that order) form a square in $\mathcal{C}(S_{1,n} - \cup c_i)$.

Now, if a_1 and b_1 are in fact adjacent with respect to $\mathcal{P}$, then they lie on a common connected component of $S_{1,n} - \cup c_i$, which must be homeomorphic to $S_{0,5}$ or $S_{1,2}$. As there are no squares in $\mathcal{C}(S_{0,5})$ or $\mathcal{C}(S_{1,2})$, it follows that there are no curves a_2 and b_2 with the property that $\{a_i, b_j, c_1, \cdots, c_{n-2}\}$ is a pants decomposition for any choice of $i, j \in \{1, 2\}$.

Since ϕ must preserve the existence of the given squares in $\mathcal{C}(S_{1,n} - \cup c_i)$, it follows that superinjectivity must preserve the property of nonadjacency. $\square$

To simplify the proofs, we will make the assumption of $n \geq 4$ in Lemmas 12 through 14. After Lemma 14, we explain how the argument should be adjusted for the case of $n = 3$.

Linear and cyclic pants decompositions. Our next goal is to show that ϕ preserves the topological types of curves. The first step is to give a classification of pants decompositions of $S_{1,n}$. Below, a *linear* pants decomposition is one which is topologically equivalent to the one on the left hand side of Figure 1 and a *cyclic* pants decomposition is one which is equivalent to the one on the right. Note that since $n \geq 3$, the adjacency graphs are distinct.
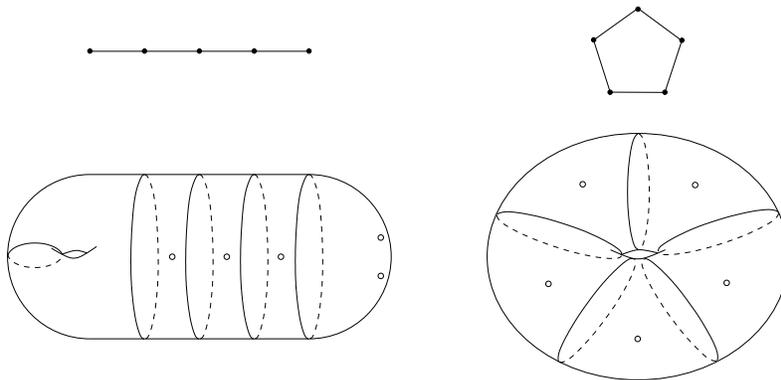

FIGURE 1. Linear (left) and cyclic (right) pants decompositions for $S_{1,5}$.

We shall call the closures of the components of $S_{1,n} - \mathcal{P}$ the *pairs of pants* of $\mathcal{P}$.

Lemma 12. *Let $n \geq 4$. If $\mathcal{P}$ is a pants decomposition of $S_{1,n}$ with the property that $\mathcal{G}(\mathcal{P})$ does not contain a triangle, then $\mathcal{P}$ is either a linear pants decomposition or a cyclic pants decomposition.*

Proof. None of the pairs of pants of $\mathcal{P}$ can have 3 distinct curves of $\mathcal{P}$ for their boundary, since that would form a triangle in the adjacency graph. Thus, any pair of pants of $\mathcal{P}$ is of one of the following types:

(1) torus with one boundary
(2) punctured annulus
(3) twice-punctured disk

We imagine constructing $S_{1,n}$ out of these pieces. The only surfaces without boundary which can be built from pairs of pants of types 1 and 3 are $S_{2,0}$, $S_{0,4}$, and $S_{1,2}$. Thus, we may conclude that $\mathcal{P}$ has at least one pair of pants of type 2. Moreover, all of the type 2 pairs of pants of $\mathcal{P}$ lie in a connected chain, since each pair of pants of type 1 or 3 is connected to only one other pair of pants.

If we add a type 1 pair of pants to each end of this chain of type 2 pairs of pants, then the result is $S_{2,m}$ for some m. If we cap off the chain by a two pairs of pants of type 3, the result is $S_{0,m}$ for some m. We can attach one type 1 and one type 3 pair of pants to the ends of the chain—this yields a linear pants decomposition of $S_{1,n}$. If we fail to attach a pair of pants to either end of this chain, we get a surface with boundary, so the only other possibility is to attach the ends of the chain together—this gives a cyclic pants decomposition of $S_{1,n}$.   □

A *side* of a separating curve z is a connected component of $S_{1,n} - z$. In the following lemma, when we say ϕ *preserves sides* of a separating curve z, we mean that if a and b are curves on the same side of z, then $\phi(a)$ and $\phi(b)$ are on the same side of $\phi(z)$.

Lemma 13. *Let $n \geq 4$. ϕ preserves the topological types of curves, and ϕ preserves sides of separating curves.*

Proof. First, if a is a nonseparating curve in $S_{1,n}$, then a fits into a cyclic pants decomposition $\mathcal{P}$ of $S_{1,n}$. By Lemmas 11 and 12, $\phi(\mathcal{P})$ is cyclic. Since all curves in a cyclic pants decomposition are nonseparating, $\phi(a)$ must be nonseparating.

Now if a is any curve in $S_{1,n}$, then a fits into a linear pants decomposition $\mathcal{P}$ of $S_{1,n}$. Again, by Lemmas 11 and 12, $\phi(\mathcal{P})$ is linear, and ϕ induces an isomorphism of $\mathcal{G}(\mathcal{P})$ with $\mathcal{G}(\phi(\mathcal{P}))$. We have already shown that ϕ preserves nonseparating curves, so ϕ must take the unique vertex of $\mathcal{G}(\mathcal{P})$ representing a nonseparating curve to the unique vertex of $\mathcal{G}(\phi(\mathcal{P}))$ representing a nonseparating curve. Since the rest of the graph isomorphism is determined, it follows that ϕ preserves the topological type of every curve in $\mathcal{P}$, in particular a.

It follows from the above argument that ϕ preserves sides of separating curves.   □

Duality. By a *k-curve*, we will mean a separating curve in $S_{1,n}$ with exactly k punctures on its genus 0 side. Denote by $\mathcal{N}$ the set of all nonseparating curves and 2-curves in $S_{1,n}$.

We call a pair of curves a and b of $\mathcal{N}$ *dual* if they are nonseparating and $i(a,b) = 1$ or if they are 2-curves and $i(a,b) = 2$.

To show that duality is preserved by ϕ, we will require the following vocabulary: let $\mathcal{N}'$ denote the union of the 3-curves and n-curves in

$S_{1,n}$. A *small* side of an element z of $\mathcal{N}'$ is a side of z homeomorphic to $S_{1,1}$ (in the case z is an n-curve) or $S_{0,4}$ (in the case z is a 3-curve). Note that when $n = 3$, any element of $\mathcal{N}'$ has two small sides.

Lemma 14. *For $n \geq 4$, ϕ preserves duality.*

Proof. Two curves a and b are dual if and only if there exists an element z of $\mathcal{N}'$ and curves x and y in $S_{1,n}$ so that: a and b lie on the same small side of z, x intersects z and a but not b, and y intersects z and b but not a. One direction is proven by construction. The other direction is proven for nonseparating curves by Ivanov and for 2-curves by Bell–Margalit [19] [3] (see also [25] [22]).

Now, note that the argument of Lemma 13 shows that ϕ preserves small sides of curves. Thus, ϕ preserves all of the properties used in this characterization of duality, and so the lemma follows. $\qquad\square$

The case $n = 3$. The subtle point for $S_{1,3}$ is that there are two topological types of pants decompositions whose adjacency graphs are triangles: those with either 2 or 3 nonseparating curves. Thus, our argument in the proof of Lemma 13 that ϕ preserves nonseparating curves does not work. However, since every pants decomposition of $S_{1,3}$ with a linear adjacency graph is linear in the sense of Lemma 12, the argument of Lemma 13 does still show that ϕ preserves the sets $\mathcal{N}$ and $\mathcal{N}'$ (and sides of elements of $\mathcal{N}'$). In that case, the argument of Lemma 14 shows that ϕ preserves duality in the case of $n = 3$ (even though we don't know topological types are preserved!). Now, we have the following distinction between nonseparating curves and 2-curves: given a nonseparating curve c, we can find three disjoint nonseparating curves which are all dual to c. However, a 2-curve (on any surface) can only be dual to at most 2 disjoint curves. Thus ϕ preserves nonseparating curves, and it then follows as in the proof of Lemma 13 that ϕ preserves 2-curves.

We remark that the above argument works in general for any $n \geq 3$. However, it is more complicated than necessary when $n > 3$..

We now return to the assumption that $n \geq 3$. Below, when we refer to Lemmas 13 and 14, we will mean the inclusive statements for all $n \geq 3$.

Triangulations. We define a *triangle* to be a triple of pairwise dual 2-curves in $S_{1,n}$ which lie on the genus 0 side of a 3-curve. We may think of a triangle as an ideal triangle in the usual sense by replacing

each 2-curve with the unique arc connecting the punctures on the genus 0 side of the 2-curve. Note that the arcs corresponding to two 2-curves are disjoint whenever the 2-curves are disjoint or dual.

A *2-curve ideal triangulation* of $S_{1,n}$ is a collection of 2-curves whose corresponding arcs form an ideal triangulation of $S_{1,n}$ (in the usual sense). It is not hard to construct 2-curve ideal triangulations of $S_{1,n}$ when $n \geq 3$.

It follows immediately from Lemma 13, Lemma 14, the injectivity of ϕ, and the definition of superinjectivity that ϕ preserves triangles and 2-curve ideal triangulations. If $\mathcal{T}$ is a 2-curve ideal triangulation of $S_{1,n}$, it follows that $\phi(\mathcal{T})$ is a 2-curve ideal triangulation of $S_{1,n}$ which is abstractly isomorphic to $\mathcal{T}$. We thus have:

Lemma 15. *If $\mathcal{T}$ is a 2-curve ideal triangulation of $S_{1,n}$, then there exists an element f of $\mathrm{Mod}(S_{1,n})$ for which $f(\mathcal{T}) = \phi(\mathcal{T})$.*

To show that f agrees with ϕ on the rest of $\mathcal{C}(S_{1,n})$, we will make use of the rigid combinatorial structure of the arc complex, which we now define.

Arc complex. The *arc complex* $\mathcal{A}(S)$ is the abstract simplicial flag complex with a vertex for each arc in the surface S and edges corresponding to disjointness. Note that 2-curve ideal triangulations correspond to maximal simplices of $\mathcal{A}(S)$.

It is not hard to see that any codimension 1 simplex of $\mathcal{A}(S)$ is contained in at most two simplices of maximal dimension. Also, there is a well-known theorem that $\mathcal{A}(S)$ is *chain connected*, which means that any two maximal simplices are connected by a sequence of maximal simplices, where consecutive simplices in the sequence share a codimension 1 face. There are various proofs of this theorem in the literature, by Mumford (see Harer's paper [12]), Penner [26], Bowditch–Epstein [6], Mosher [24], and Hatcher [14]. Several of these papers credit the idea of this theorem to W. Thurston (see [1]).

From these facts about $\mathcal{A}(S)$, it follows that:

Lemma 16. *Any injective simplicial map $\mathcal{A}(S) \to \mathcal{A}(S)$ is determined by its action on a single maximal simplex.*

We already have that ϕ agrees with some element f of $\mathrm{Mod}(S_{1,n})$ on a maximal simplex of $\mathcal{A}(S_{1,n})$, namely any 2-curve ideal triangulation. If

we show that ϕ extends to an injective simplicial map on all of $\mathcal{A}(S_{1,n})$, it will follow from Lemma 16 that ϕ agrees with f on all of $\mathcal{A}(S_{1,n})$. From there, it will be easy to show that ϕ agrees with f on $\mathcal{C}(S_{1,n})$.

Induced map on $\mathcal{A}(S_{1,n})$. There is a natural way to extend ϕ to a map, which we also call ϕ, on the vertices of $\mathcal{A}(S_{1,n})$. If α is an arc in $S_{1,n}$, we call the set of nontrivial boundary curves of a regular neighborhood of α the *pushoffs* of α.

If α is a nonseparating arc starting and ending at the same puncture, or a separating arc with two pushoffs, then it follows from Lemmas 11, 12, and 13 that the images under ϕ of the pushoffs of α bound a punctured annulus. Define $\phi(\alpha)$ to be the unique arc in this annulus (note $\phi(\alpha)$ is topologically equivalent to α).

If α is an arc with a single pushoff, then this pushoff must be a 2-curve. In this case, $\phi(\alpha)$ is determined by the image of its pushoff and the following natural action of ϕ on the punctures of $S_{1,n}$: the puncture "shared" by a pair of dual 2-curves is sent to the puncture shared by the images under ϕ of those 2-curves (we are using Lemma 14).

In his thesis, Korkmaz gives a proof that this action on the punctures of $S_{1,n}$ is well-defined [21]. He further proves:

Lemma 17. *ϕ is a simplicial map of $\mathcal{A}(S_{1,n})$.*

Both arguments hold in our setting, since they only use properties of superinjective maps which we have already established.

It follows easily from the injectivity of the action of ϕ on curves that:

Lemma 18. *The action of ϕ on arcs is injective.*

Now, we can finally show:

Proposition 19. *ϕ is induced by f.*

Proof. It follows from Lemmas 15 through 18 that ϕ agrees with f on all of $\mathcal{A}(S_{1,n})$, and hence ϕ and f agree on all 2-curves. It now follows that ϕ and f agree on all of $\mathcal{C}(S_{1,n})$. Indeed, given any curve c in $S_{1,n}$, we can *fill $S_{1,n} - c$* with 2-curves in the sense that c is the unique curve disjoint from the collection of 2-curves. Since ϕ and f both preserve disjointness, it follows that f and ϕ agree on c. $\square$

This completes the proof of Theorem 4.

Remarks. Our techniques give a particularly streamlined proof of the theorem of Bell–Margalit that any superinjective map of $\mathcal{C}(S_{0,n})$ is induced by $\mathrm{Mod}(S_{0,n})$ when $n \geq 5$. Indeed, for a genus 0 surface, a "linear" pants decomposition consists of a chain of punctured annuli with a twice-punctured disk on each end. It follows then from Lemma 11 that a superinjective map preserves the topological types of curves. In their proof, this is the main difficult step.

We also note that the corresponding theorem of Irmak for $S_{2,0}$ follows immediately since, by the same argument of Luo used in Section 4 to show $\mathcal{C}(S_{1,2}) \cong \mathcal{C}(S_{0,5})$, we have that $\mathcal{C}(S_{2,0}) \cong \mathcal{C}(S_{0,6})$. It then follows from the result of Bell–Margalit that any superinjective map of $\mathcal{C}(S_{2,0})$ is an automorphism, and Ivanov proved that any automorphism of $\mathcal{C}(S_{2,0})$ is induced by an element of $\mathrm{Mod}(S_{2,0})$.

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