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Wed 05/10/06

PERRON'S METHOD & EXISTENCE RESULT

Consider the Dirichlet problem $\begin{cases} F(x, u, Du, D^2u) = 0, & \Omega. \\ u = 0, & \partial\Omega. \end{cases}$ (DP)

- $u \in USC(\bar{\Omega})$ is a vis. sub/super sol. of (DP) if u is a vis. sub/super sol. of $F=0$ in Ω and $u \leq / \geq 0$ on $\partial\Omega$.
- F is assumed to be proper and continuous.

Define for each $u : \Omega \to [-\infty, \infty]$

$$u^*(x) = \lim_{r \downarrow 0} \sup\{u(y) : y \in \Omega \text{ and } |y-x| \leq r\}$$

$$u_*(x) = \lim_{r \downarrow 0} \inf\{u(y) : y \in \Omega \text{ and } |y-x| \leq r\}$$

u^* is called the upper semi-cont envelope of u, i.e. u^* is the smallest USC function such that $u \leq u^*$.

Theorem 6 (Perron's Method)

Let comparison hold for (DP), i.e. theorem 5 is satisfied.

Suppose there exists a vis. subsol $\underline{u}$ and a vis. super sol. $\bar{u}$ of (DP) such that $\underline{u}_*(x) = \bar{u}_*(x) = 0$ for $x \in \partial\Omega$.

Then

$$W(x) = \sup\{w(x) : \underline{u} \leq w \leq \bar{u} \text{ and } w \text{ is a vis. sub sol. of (DP)}\}$$

is a vis. sol. of (DP).

Convention: (sub/super) solution means vis. (sub/super) sol.

Lemma 7

Let $\mathcal{G}$ be a family of sub sols of F in Ω.

Let $w(x) = \sup\{u(x) : u \in \mathcal{G}\}$.

Then w^* is a sub sol of $F \leq 0$ in Ω.

Let u be a sub. sol. of $F \leq 0$ such that u_* is not a super sol.

We can assume at $0 \in \Omega$ that

$F(0, u_*(0), p, X) < 0$

for some $(p, X) = (D\varphi(0), D^2\varphi(0))$ where $\varphi \in C^2$ such that $\varphi(0) = u_*(0)$

and $u_*(x) > \varphi(x)$, $\forall x \neq 0$.

It is clear that

$$u_*(x) > u_*(0) + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle + o(|x|^2) \text{ for } x \to 0. \qquad (11)$$

By the continuity of F

$u_{\delta,\gamma}(x) = u_*(0) + \delta + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle - \gamma|x|^2$

is a classical sol. of $F \leq 0$ in $B_r = \{|x| < r\}$ for $r, \delta, \gamma > 0$ small enough.

Since $u(x) \geq u_*(x)$ and (11), if we choose

$$\delta = \left(\frac{r^2}{8}\right)\gamma$$

then $u(x) > u_{\delta,\gamma}(x)$, for $\frac{r}{2} \leq |x| \leq r$, if r sufficiently small.

Then by the lemma 7

$$U(x) = \begin{cases} \max\{u(x), u_{\delta,\gamma}(x)\} & \text{if } |x| < r, \\ u(x), & \text{otherwise}. \end{cases} \qquad (12)$$

is a sub sol. of $F = 0$ in Ω. (Note that $U = U^*$).

We observe that by the definition of u_* there exists a sequence

$(x_n, u_n(x_n)) \to (0, u_*(0))$ and $\lim_{n\to\infty} (U(x_n) - u(x_n)) = u_{\delta,\gamma}(0) - u_*(0) = \delta > 0.$

<u>Lemma 8</u> Let u be a sub sol. of $F = 0$ in Ω

If u_* fails to be a super sol. at some point $\hat{x}$, then for any small $\varepsilon > 0$

there is a sub sol. U_ε of $F = 0$ in Ω satisfying

$$\begin{cases} U_\varepsilon(x) \geq u(x), & \sup_\Omega (U_\varepsilon - u) > 0 \\ U_\varepsilon(x) = u(x) & \text{for } |x - \hat{x}| \geq \varepsilon. \end{cases}$$

Proof of theorem 6

We note that

$\underline{u}_* \leqslant W_* \leqslant W \leqslant W^* \leqslant \overline{u}^*$

and $W_* = W = W^* = 0$ on $\partial\Omega$.

By lemma 7, $W^*(x)$ is a sub sol. of (DP)

and by comparison $W^* \leqslant \overline{u}$ in $\overline{\Omega}$. Hence, $W = W^*$.

If W_* fails to be a super sol. at some point $\hat{x} \in \Omega$,

let W_ε be provided by lemma 8, then

$\underline{u} \leqslant W_\varepsilon$ and $W_\varepsilon = 0$ on $\partial\Omega$.

By comparison $W_\varepsilon \leqslant \overline{u}$ and thus $W_\varepsilon \leqslant W$, a contradiction.

We conclude that W_* is a super sol and then by comparison

$W^* = W \leqslant W_*$.

Therefore W is continuous and is a solution of (DP).

The uniqueness of W follows from comparison.

LIMIT OPERATIONS WITH VISCOSITY SOLUTIONS

Let u_n be a sequence of vis. subsolutions of

$$F_n(x, u, Du, D^2u) = 0, \quad \Omega.$$

Define
$$\overline{U}(z) = \lim_{j\to\infty} \sup\{u_n(x) : n \geq j,\ x \in \Omega \text{ and } |z-x| < 1/j\}$$

(the "lim sup" operation and the * operation are performed simultaneously)

<u>Theorem 9</u>

If $G(x,r,p,X) = \lim_{n\to\infty} F_n(x,r,p,X)$ and $\overline{U}(z) < \infty$, $z \in \Omega$,

then $\overline{U}$ is a vis. subsolution of $G = 0$.

<u>Example</u>

Let u_n be the classical solution of

$$\begin{cases} -\operatorname{div}\left[(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}} \nabla u\right] = f, & \Omega \\ u = 0, & \partial\Omega. \end{cases}$$

If p is large enough, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$

and thus $u_n \longrightarrow u$ uniformly in $C^\alpha(\overline{\Omega})$.

It is easy to see that u solves

$$\begin{cases} -\Delta_p u = f, & \Omega \\ u = 0, & \partial\Omega \end{cases} \tag{13}$$

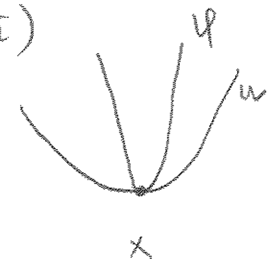
in the weak sense.

However, by theorem 9, u is a vis solution of (13).

General boundary conditions

$$(BVP) \begin{cases} F(x,u,Du,D^2u) = 0 \quad , \Omega & (E) \\ B(x,u,Du,D^2u) = 0 \quad , \partial\Omega & (BC) \end{cases}$$

Formally, you can define a function $u \in USC(\bar{\Omega})$ a vis. subsolution of (BVP) if



$$F(x, \varphi(x), D\varphi(x), D^2\varphi(\hat{x})) \leq 0 \quad , x \in \Omega, \qquad (14)$$
$$B(x, \varphi(x), D\varphi(x), D^2\varphi(\hat{x})) \leq 0 \quad , x \in \partial\Omega.$$

Example

Consider, for $\varepsilon > 0$,
$$\begin{cases} -\varepsilon u'' + u' + u = x+1 \quad , x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

With $B(1,r,p) = p$ and $B(0,r,p) = -p$ we obtain $\frac{\partial u}{\partial n} = 0$.

Then $u_\varepsilon = x + \dfrac{e^{\lambda_0} - 1}{\lambda_1(e^{\lambda_1} - e^{\lambda_0})} e^{\lambda_1 x} + \dfrac{1 - e^{\lambda_1}}{\lambda_0(e^{\lambda_1} - e^{\lambda_0})} e^{\lambda_0 x}$,

where $\lambda_{1,0} = \frac{1}{2\varepsilon}\left(1 \pm \sqrt{1+4\varepsilon}\right)$.

The limit $\lim_{\varepsilon\to 0} u_\varepsilon = u = x + e^{-x}$ uniformly on $[0,1]$

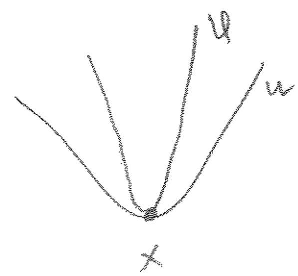
since $\lambda_1 \to \infty$ and $\lambda_0 \to -1$ as $\varepsilon \to 0$.

The function $u = x + e^{-x}$ satisfies
$$u' + u = x+1, \text{ in } (0,1) \quad \text{and} \quad u'(0) = 0$$

BUT $u'(1) = 1 - 1/e > 0$.

Therefore u is not a solution of (14) with $B(1, u'(1)) = 0$.

However, at $x=1$, $\min\{u' + u - (x+1), u'\} \leq 0$.



Definition

u is a vis. subsol of BVP if

$F(x, \varphi, D\varphi, D^2\varphi) \leq 0$ for $x \in \Omega$

$\min\{F(x,\varphi,D\varphi,D^2\varphi), B(x,\varphi,D\varphi,D^2\varphi)\} \leq 0$ for $x \in \partial\Omega$.

PARABOLIC PROBLEMS

$$\begin{aligned}
&(E) && u_t + F(t,x,u,Du,D^2u) = 0 \quad \text{in} \quad (0,T)\times\Omega, \\
&(BC) && u(t,x) = 0 \quad \text{for} \quad 0 \leq t < T \text{ and } x \in \partial\Omega, \qquad (15)\\
&(IC) && u(0,x) = \psi(x) \quad \text{for} \quad x \in \overline{\Omega}.
\end{aligned}$$

$$\overline{F} : \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R}^{N+1} \times S(N) \to \mathbb{R}$$
$$((t,x), r, (q,p), X) \longmapsto q + F(t,x,r,p,X).$$

$\Omega_T = (0,T)\times\Omega.$

A vis. subsol. of (E) is defined to be a vis. subsol. of $\overline{F} = 0$ in Ω_T.

We call u is a vis. subsol. of (15) if

- $u \in USC([0,T)\times\overline{\Omega})$ such that u is a vis. subsol. of (E)
- $u(t,x) \leq 0$ for $0 \leq t < T$ and $x \in \partial\Omega$.
- $u(0,x) \leq \psi(x)$ for $x \in \overline{\Omega}$.