Some loose ends

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1. Lebesgue integration on $\mathbb{R}^n$

If $f(x_1, \dots, x_n)$ is a smooth function on $\mathbb{R}^n$ with support in the open set X, its integral is

$$\int_X f(x)\, dx_1 \dots dx_n \,,$$

which can be explicitly calculated (rarely) by reducing it to one-dimensional integrals, where one can apply the fundamental theorem of calculus. If we make a change of variables $x = h(y)$ where h is an invertible smooth function the integral becomes

$$\int_{h^{-1}(X)} f\big(h(y)\big)\, |\partial x/\partial y|\, dy$$

since $x \in X$ if and only if $y = h^{-1}(x)$ lies in $h^{-1}(X)$. What is important here is that this formula involves the absolute value of the Jacobian determinant.

The change of variables formula in 1D might seem a bit paradoxical but it agrees with the usual rules of calculus. For example

$$\int_{-\infty}^{\infty} f(x)\,dx = -\int_{\infty}^{-\infty} f(-y)\,dy = \int_{-\infty}^{\infty} f(-y)\,dy$$

The point is that this integral represents an integral of a measure.

2. Integration of forms on $\mathbb{R}^n$

If ω is an n-form on $\mathbb{R}^n$ it can be written as $f(x)\,dx_1 \wedge \ldots \wedge dx_n$ and then its integral is

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x)\,dx_1 \ldots dx_n$$

The point is that we have to first arrange the formula for ω so as to match the standard orientation of $\mathbb{R}^n$.

3. Integration on oriented manifolds

Suppose M to be an oriented manifold. We can cover it by coordinate patches U_i embedded in $\mathbb{R}^n$ in such a way that the orientations all match that of M, and we can find a partition of unity φ_i subordinate to this covering. Then $\varphi_i \omega$ may be identified with a compactly supported form ω_i on $\mathbb{R}^n$ and

$$\int_M \omega = \sum_i \int_{U_i} \omega_i \, .$$

4. Integration on arbitrary manifolds

Suppose now that M is an arbitrary manifold of dimension n. At each point m of M we have the one-dimensional real vector space $\bigwedge^n T_x$. The fibre bundle $\widetilde{M}$ of orientations on M is the quotient of $\bigwedge^n T_x - \{0\}$ by the positive real numbers, a set of two elements. The space $\widetilde{M}$ is a two-fold covering of M. The manifold M is orientable if and only if this bundle has a section, which is to say that at each point we have a continuous choice of orientation. If it is orientable then we can integrate forms over M, but only after making a choice of orientation. Reversing the orientation will change the sign of the integral. So there is no canonical way to integrate forms on M.

There is, however, a canonical way to integrate something else, called a density or twisted n-form.

The covering $\widetilde{M}$ has a conical involution, interchanging orientations at any point of M. The n-forms on M may be identified with forms on $\widetilde{M}$ that are invariant under this involution. Since changing orientation changes the sign of an integral of a form, the integral of such a form on $\widetilde{M}$ is 0. A twisted n form on M is defined to be an n-form on $\widetilde{M}$ that is taken to its negative by the involution. If $\widetilde{\omega}$ is such a form on $\widetilde{M}$ then by definition

$$\int_M \widetilde{\omega} = \frac{1}{2} \int_{\widetilde{M}} \widetilde{\omega}$$

In other words, what is invariantly defined on an arbitrary manifold is the integral of a twisted n-form.

The twisted n-forms on a manifold are sections of a one-dimensional fibre bundle on M. The fibre at x is the space of all maps f from $\bigwedge^n T_x$ to $\mathbb{R}$ such that

$$f(cv) = |c| f(v)$$

On any manifold there always exists at least one twisted n-form that never vanishes.

5. Homogeneous fibre bundles

Suppose now that G is a Lie group and H a closed subgroup. If (σ, U) is a finite-dimensional representation of H, thne there is associated to it a fibre bundle over $H\backslash G$ whos e fibre at any point is non-canonically equal to U. Geometrically it is the quotient of $U \times G$ by the group H taking (u, g) to $(\sigma(h)u, hg)$. The sections of this bundle over $H\backslash G$ are the functions

$$f\colon G \longrightarrow U$$

such that $f(hg) = \sigma(h)f(g)$ for all h in H and g in G. One representation of H is that on the tangent space at 1 of $H\backslash G$, which may be identified with $\mathfrak{h}\backslash\mathfrak{g}$. The bundle to conjugation Ad is the tangent bundle. Another is the one dimensional representation of H taking

$$h \longrightarrow |\det \mathrm{Ad}_{\mathfrak{h}\backslash\mathfrak{g}}(h)|^{-1}$$

and the associated bundle is twisted n-forms.

Take $G = \mathrm{SL}_2(\mathbb{R})$ and $H = P$. Here $\mathfrak{p}\backslash\mathfrak{g} = \overline{\mathfrak{n}}$ and the twisted n-forms correspond to the character

$$\delta_P\colon \begin{bmatrix} a & x \\ 0 & 1/a \end{bmatrix} \longmapsto a^2$$

Since $a^2 > 0$ these do not differ from ordinary n-forms. This remains true for the spaces $\mathbb{P}^1(\mathbb{R}^n)$ with n odd, but fails for the non-orientable cases with n even.

At any rate, a smooth real twisted n-form on $P\backslash G$ may be identified a smooth function f from G to $\mathbb{R}$ such that $f(pg) = \delta_P(p)f(g)$. I write integration of twisted n-forms as

$$\int_{P\backslash G} \omega$$

Since $G = PK$, the quotient $P\backslash G$ may be identified with $K \cap P\backslash K$, and if we assign K a total measure 1 integration on $P\backslash G$ may be identified with integration over K.

There is another way to put this. If f is a smooth function of compact support on G, then

$$\overline{f}(g) = \int_P f(pg)\, d_\ell p$$

is a density on $P\backslash G$—$\overline{f}(pg) = \delta_P(p)\overline{f}(g)$. Then with suitable normalizations

$$\begin{aligned}\int_G f(g)\, dg &= \int_{P\backslash G} \overline{f}(x) \\ &= \int_K dk \int_P f(pk)\, d_\ell p \\ &= \int_K dk \int_A \delta_P(a)^{-1}\, da \int_N f(nak)\, dn\end{aligned}$$

since the integral with respect to $d_\ell P$ can also be expressed as

$$\int_A \delta_P(a)^{-1}\, da \int_N f(na)\, dn\,.$$

There is another formula for integration over $P\backslash G$. The set $P\overline{N}$ is open in G, and the integral

$$\int_{\overline{N}} f(\overline{n})\, d\overline{n}$$

converges. It is, up to a constant, another valid formula. If we identify $\overline{N}$ with $\mathbb{R}$, what is the constant?

6. The smooth principal series

Any character (continuous homomorphism into $\mathbb{C}^\times$) of A is of the form

$$\chi_{s,m}\colon \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} \longmapsto |x|^s \operatorname{sgn}^m(x)$$

for some s in $\mathbb{C}$ and $m = 0, 1$. This will be a unitary character if and only if $s = it$ for some real number t.

Any character of A determines one of P since $P/N = A$. Any continuous irreducible representation of P is of this form (in particular trivial on N). In any continuous finite-dimensional representation of P the subgroup N is taken to unipotent matrices.

The principal series representations of G are those induced from characters of P.

$$\mathrm{Ind}^\infty(\chi \,|\, P, G)$$
$$= \{f \in C^\infty(G, \mathbb{C}) \,|\, f(pg) = \delta_P^{1/2}\chi(p)f(g) \text{ for all } p \in P, g \in G\}$$

The group G acts by the right regular action:

$$R_g f(x) = f(xg)$$

- $\mathrm{Ind}^\infty(\delta_P^{-1/2}) = C^\infty(P\backslash G)$
- $\mathrm{Ind}^\infty(\delta_P^{+1/2}) = \Omega^\infty(P\backslash G)$
- $\mathrm{Ind}^\infty(\chi^{-1}) =$ **the dual of** $\mathrm{Ind}^\infty(\chi)$

$$\langle f, \varphi \rangle = \int_{P\backslash G} f(x)\varphi(x)\, dx$$

- $\mathrm{Ind}^\infty(\chi)$ **is unitary if** χ **is.**

The best way to picture $\operatorname{Ind}^\infty(\chi)$ is to describe its restriction to K.

Restricting f to K determines a map from K to $\mathbb{C}$ such that

$$f(pk) = \chi(p)f(k)$$

for all p in $P \cap K$. Because $G = PK$ this is an isomorphism. Since $P \cap K = \pm I$ and $\chi(-I) = (-1)^m$:

$$\operatorname{Ind}^\infty(\chi)\,|K = \widehat{\sum}_{n \equiv m \bmod 2} \varepsilon^n$$

where $\widehat{\sum}$ means a topological sum (C^∞ Fourier series).

Let

$$\varphi_n(pk) = \delta^{1/2}\chi(p)\varepsilon^n(k)$$

If

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $g = pk$ **where**

$$p = \begin{bmatrix} 1/r & (ac+bd)/r \\ 0 & r \end{bmatrix} \quad (r = \sqrt{c^2+d^2})$$

$$k = \begin{bmatrix} \gamma & -\sigma \\ \sigma & \gamma \end{bmatrix} \quad (\gamma = d/r, \sigma = c/r)$$

Therefore

$$\varphi_n(g) = \delta^{1/2}\chi(1/r)(\gamma + i\,\sigma)^n$$

7. Explicit formulas

The Lie algebra $\mathfrak{g}$ acts on the subspace of finite sums of the φ_n. Recall the basis of the complex Lie algebra

$$\kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$x_+ = \begin{bmatrix} 1 & -i \\ -i & 0 \end{bmatrix}$$

$$x_- = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}$$

$$[\kappa, x_\pm] = \pm 2i\, x_\pm$$

so that

$$\begin{aligned} \kappa\,\varphi_n &= ni\,\varphi_n \\ \kappa(x_\pm\varphi_n) &= x_\pm(\kappa\varphi_n) \pm 2ix_\pm\varphi_n \\ &= (n \pm 2)i\,(x_\pm\varphi_n) \\ x_\pm\,\varphi_n &= \textit{constant} \cdot \varphi_{n\pm 2} \end{aligned}$$

$$x_\pm \varphi_n = \mathit{constant} \cdot \varphi_{n\pm2}$$

What is the constant? **Since** $\varepsilon_n(1) = 1$

$$x_\pm \varphi_n(1) = \mathit{constant}$$

Here the Lie algebra acts on the right. So we use the basic trick (seen before).

$$R_X f(g) = L_{gXg^{-1}} f(g)$$

here with $g = 1$**. Since**

$$x_\pm = \alpha \mp i(\kappa + 2\nu_+)$$

$$\begin{aligned} R_{x_\pm}\varepsilon_n(1) &= [L_{\alpha\mp2i\nu_+} \mp R_{i\kappa}]\varepsilon_n(1) \\ &= (s + 1 \pm n) \end{aligned}$$

Summary:

$$\kappa\,\varepsilon_n = ni\,\varepsilon_n$$

$$x_\pm\,\varepsilon_n = (s+1\pm n)\varepsilon_n$$

We have seen this before when $s = -1$ and $s+1 = 0$ (except for some small change of sign) caused by a difference between left and right actions. The space of Harmonic functions is isomorphic to $\mathrm{Ind}(\delta^{-1/2})$. More generally:

> **Every irreducible $(\mathfrak{g}, K)$-representation can be embedded into a principal series representation.**

To be proven in a later lecture.

8. Intertwining operators

Some principal series are isomorphic to other principal series. Some principal series are reducible. To figure out what's going on, we need to calculate the G-covariant (or $(\mathfrak{g}, K)$-covariant) maps from one principal series to another.

The start is a version of Frobenius reciprocity. I recall what this says for a finite group. Let H be a subgroup of another group G. If σ is an irreducible representation of H, we want to know how often an irreducible representation π of G occurs in the representation $I(\sigma)$ induced by σ. The answer is that π occurs as often in $I(\sigma)$ as σ occurs in the restriction of π to H:

$$\dim \mathrm{Hom}_G\big(\pi, I(\sigma)\big) = \dim \mathrm{Hom}_H(\sigma, \pi)$$

But since represenations of finite groups always decompose completely, this is also

$$\dim \mathrm{Hom}_H(\pi, \sigma)$$

Theorem. **(Frobenius reciprocity for finite groups)** *Suppose $H \subseteq G$ are finite groups. If (σ, U) is any finite dimensional representation of H and (π, V) is one of G then there is a canonical isomorphism*

$$\mathrm{Hom}_G(\pi, I(\sigma)) \cong \mathrm{Hom}_H(\pi, \sigma)$$

$$I(\sigma) = \{f\colon G \to U \,|\, f(hg) = \sigma(h)\}$$

Either side determines the other—$F_G(v) = F_H\big(\pi(g)v\big)$.

Let

$$\Lambda_1\colon \operatorname{Ind}^\infty(\chi) \longrightarrow \mathbb{C}, \quad f \longmapsto f(1)$$

Theorem. **(Frobenius reciprocity for principal series)** *If V is a smooth representation of G then composition with Λ_1 induces an isomorphism*

$$\operatorname{Hom}\big(V, \operatorname{Ind}^\infty(\chi \,|\, P, G)\big) = \operatorname{Hom}_P(V, \delta^{1/2}\chi)$$

The Lie algebra $\mathfrak{n}$ acts trivially on $\mathbb{C}$, so any P-map from V to $\delta^{1/2}\chi$ takes $\nu_+ v$ to 0. It must annihilate the subspace $\mathfrak{n}V$ of V spanned all the $\nu_+ v$. In other words it must factor through the quotient $V/\mathfrak{n}V$, on which A acts. So a new version of the theorem is

$$\begin{aligned}\operatorname{Hom}\big(V, \operatorname{Ind}^\infty(\chi \,|\, P, G)\big) &= \operatorname{Hom}_A(V/\mathfrak{n}V, \delta^{1/2}\chi) \\ &= \operatorname{Hom}_A\big(\chi^{-1}\delta^{-1/2}, \widehat{V}[\mathfrak{n}]\big)\end{aligned}$$

There are two kinds of N-invariant functionals on $\mathrm{Ind}^\infty(\chi)$, corresponding to the two components in the Bruhat decomposition

$$G = P \cup PwN$$

Formally, we have the integral

$$\Lambda_w(f) = \int_N f(wn)\, dn$$

which satisfies

$$\begin{aligned} \Lambda_w(R_{n_*} f) &= \int_N f(wnn_*)\, dn \\ &= \Lambda_w(f) \end{aligned}$$

...

. . . **and then**

$$\begin{aligned}\Lambda_w(R_a f) &= \int_N f(wna)\,dn \\ &= \int_N f(wa\cdot a^{-1}na)\,dn \\ &= \int_N f(waw^{-1}\cdot w\cdot a^{-1}na)\,dn \\ &= \delta^{1/2}\chi(a^{-1})\int_N f(w\cdot a^{-1}na)\,dn \\ &= \delta^{-1/2}(a)\chi^{-1}(a)\cdot\delta(a)\int_N f(wn)\,dn \\ &= \delta^{1/2}(a)\chi^{-1}(a)\Lambda_w(f)\end{aligned}$$

giving rise to a G-homomorphism

$$T_w\colon\ \mathrm{Ind}^\infty(\chi)\longrightarrow\mathrm{Ind}^\infty(\chi^{-1})$$

When is the integral

$$\Lambda_w(f) = \int_N f(wn)\,dn = \int_{\mathbb{R}} f(wn_x)\,dx \quad \left(n_x = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right)$$

defined? Since

$$wn_x = \begin{bmatrix} 1/\sqrt{x^2+1} & \cdots \\ & \sqrt{x^2+1} \end{bmatrix} \begin{bmatrix} x/\sqrt{x^2+1} & -1/\sqrt{x^2+1} \\ 1/\sqrt{x^2+1} & x/\sqrt{x^2+1} \end{bmatrix}$$

$$f(wn_x) = (x^2+1)^{-(s+1)/2} f(k_x)$$

and

$$\Lambda_w(f) = \int_{\mathbb{R}} (x^2+1)^{-(s+1)/2} f(k_x)\,dx$$

Since $(x^2+1)^{-(s+1)/2} \sim 1/x^{s+1}$ **this converges and is holomorphic for** $\mathrm{RE}(s) > 0$**.**

Explicitly

$$\Lambda_w(\varphi_0) = \int_{\mathbb{R}} (x^2+1)^{-(s+1)/2}\, dx = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}$$

since $\varphi_0(k_x) = 1$. This continues meromorphically to all of $\mathbb{C}$. Similarly

$$\Lambda_w(\varphi_1) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+2}{2}\right)}$$

Since $x_\pm$ commutes with T_w and $x_\pm \cdot \varepsilon_n = (s+1\pm n)\varepsilon_n$ we see that Λ_w is meromorphic on all of $\mathrm{Ind}(\chi)$.

In fact it is meromorphic on all of $\mathrm{Ind}^\infty(\chi)$, but we'll postpone checking that.

9. Characters

If (π, V) is any smooth representation of G and f lies in $C_c^\infty(G)$ then

$$[\pi(f)](v) = \int_G f(g)\pi(g)v\,dg$$

defines V as a module over $C_c^\infty(G)$. This is an element of the vector space of continuous linear maps from V to itself.

If V has finite dimension then $\mathrm{Hom}_{\mathbb{C}}(V, V) = \widehat{V} \otimes V$, and $\pi(f)$ would be an element of this tensor product. One can introduce a topological tensor product that allows us to make the same assertion for a large class of smooth representations, but here I'll look at the case of $V = \mathrm{Ind}^\infty(\chi \,|\, P, G)$. I shall define $\pi(f)$ as an element of

$$\mathrm{Ind}(\chi^{-1} \otimes \chi \,|\, P \times P, G \times G)\,,$$

which is in fact a topological tensor product of $\widehat{V} \,\widehat{\otimes}\, V$ when V is $\mathrm{Ind}^\infty(\chi)$.

For any f in $C_c^\infty(G)$ define

$$f_\chi(g,h) = \int_A \chi\delta_P^{-1/2}(a)\,da \int_N f(h^{-1}nag)\,dn\,,$$

a function on $G \times G$.

Proposition. *The function $f_\chi(g,h)$ lies in*

$$\mathrm{Ind}^\infty(\chi^{-1} \otimes \chi \,|\, P \times P, G \times G)$$

For example

$$\begin{aligned} f_\chi(n_* g, h) &= \int_A \chi\delta_P^{-1/2}(a)\,da \int_N f(h^{-1}na \cdot n_* g)\,dn \\ &= \int_A \chi\delta_P^{-1/2}(a)\,da \int_N f(h^{-1}n \cdot a n_* a^{-1} \cdot ag)\,dn \\ &= \int_A \chi\delta_P^{-1/2}(a)\,da \int_N f(h^{-1}nag)\,dn \\ &= f_\chi(g,h) \end{aligned}$$

and

$$\begin{aligned} f_\chi(a_* g, h) &= \int_A \chi\delta_P^{-1/2}(a)\,da \int_N f(h^{-1}na \cdot a_* g)\,dn \\ &= \int_A \chi\delta_P^{-1/2}(ba_*^{-1})\,db \int_N f(h^{-1}n \cdot bg)\,dn \\ &= \chi^{-1}\delta^{1/2}(a_*)\, f_\chi(g,h) \end{aligned}$$

If F lies in $\mathrm{Ind}^\infty(\chi^{-1}\otimes\chi\,|\,P\times P, G\times G)$ and φ in $\mathrm{Ind}^\infty(\chi)$ then for each fixed h in G the product $F(g,h)\cdot\varphi(g)$ lies in $\Omega^\infty(P\backslash G)$, and hence the integral

$$\int_{P\backslash G} F(x,h)\varphi(x)\,dx = [F(\varphi)](h)$$

is defined. The map $\varphi\mapsto F(\varphi)$ is an endomorphism of $\mathrm{Ind}^\infty(\chi)$.

If V were finite-dimensional then for any f in $\widehat{V}\otimes V$ its trace when considered as an endomorphism of V would be the image of f under the canonical pairing

$$\widehat{v}\otimes v \longmapsto \langle \widehat{v}, v\rangle$$

This remains valid here. There is a canonical $G\times G$-covariant map from $\mathrm{Ind}^\infty(\chi^{-1}\otimes\chi\,|\,P\times P, G\times G)$ to $\Omega^\infty(P\times P\backslash G\times G)$ and thence to $\mathbb{C}$ and the trace of F is its image in $\mathbb{C}$.

We can do things more concretely.

$$\begin{aligned} R_f\varphi(g) &= \int_G f(x)\varphi(gx)\,dx \\ &= \int_G f(g^{-1}y)\varphi(y)\,dy \\ &= \int_K dk \int_A \delta_P^{-1}(a)\,da \int_N \varphi(nak) f(g^{-1}nak)\,dn \\ &= \int_K \varphi(k)\,dk \int_A \sigma(a)\delta_P^{-1/2}(a)\,da \int_N f(g^{-1}nak)\,dn\,. \end{aligned}$$

The trace of R_f on $\mathrm{Ind}^\infty(\chi)$ is therefore

$$\int_A \chi\delta^{-1/2}(a)\,da \int_N \overline{f}(na)\,dn \quad \textbf{where} \quad \overline{f}(an) = \int_K f(kank^{-1})\,dk$$

The result we eventually want is this:

Theorem. *The trace of* R_f *on* $\mathrm{Ind}^\infty(\chi)$ *is*

$$\int_G f(g)\Theta_\chi(g)\,dg$$

where

$$\Theta_\chi(g) = \frac{\chi(x) + \chi^{-1}(x)}{|x - x^{-1}|}$$

if g *is conjugate to* a_x *and* 0 *otherwise.*

The point here is that the character of $\mathrm{Ind}^\infty(\chi)$ is originally defined as a distribution, but it is in fact a distribution defined by the locally summable function Θ_χ.

We want to show that

$$\int_A \chi\delta^{-1/2}(a)\,da \int_N \overline{f}(na)\,dn$$

is the same as

$$\int_{G_A} f(g)\Theta_\chi(g)\,dg$$

where $\Theta_\chi(g) = \dfrac{\chi(x)+\chi^{-1}(x)}{|x-x^{-1}|}$ **if** g **is conjugate to** a_x**.**

We can write the first as

$$\int_A \chi(a)\overline{f}_P(a)\,da \textbf{ where } f_P(a) = \delta^{-1/2}(a)\int_N f(na)\,dn$$

Because Θ is conjugation-invariant we can write the other integral as

$$\int_G f(g)\Theta(g)\,dg = \frac{1}{2}\int_A |\Delta(a)|\,\Theta(a)\,da \int_{G/A} f(xax^{-1})\,dx \quad (\textbf{Weyl})$$

$$= \frac{1}{2}\int_A |\Delta(a)|\,\frac{\chi(a)+\chi^{-1}(a)}{|\Delta(a)|^{1/2}}\,da \int_{G/A} f(xax^{-1})\,dx$$

$$= \frac{1}{2}\int_A |\Delta(a)|^{1/2}\left(\chi(a)+\chi^{-1}(a)\right)da \int_{G/A} f(xax^{-1})\,dx$$

$$= \int_A \chi(a)\,|\Delta(a)|^{1/2}\,da \int_{G/A} f(xax^{-1})\,dx\,.$$

Here $\Delta(a_x) = |x - x^{-1}|$**.**

We want to show that

$$\int_A \chi(a)\overline{f}_P(a)\,da = \int_A \chi(a)\,|\Delta(a)|^{1/2}\,da \int_{G/A} f(xax^{-1})\,dx$$

i.e.

$$\delta^{-1/2}(a)\int_N dn \int_K f(knak^{-1})\,dk = |\Delta(a)|^{1/2}\int_{G/A} f(xax^{-1})\,dx$$

This depends on a lemma of Harish-Chandra's—*for any a_x in A with $x^2 \neq 1$ the transformation $n \mapsto n \cdot ana^{-1}$ is bijective with modulus $|\det \mathrm{Ad}_{\mathfrak{n}}(a) - 1| = |x^2 - 1|$.*

You'll need to know that $|x^2 - 1| = |x||x - x^{-1}| = \delta^{1/2}(a_x)\Delta(a)$**.**

The End