Classical Problems in Commutative Algebra

VIGRE Mini-course
University of Utah
7th - 18th June 2004

Homological Algebra, the Frobenius Endomorphisms and Smoothness

Claudia Miller
cmille06@syr.edu

Notes by Ananthnarayan H.
ananth@math.ukans.edu

§ 1 *Tor and Ext*

Notations and Conventions:

1. R is a commutative Noetherian ring, and M and N are finitely generated R-modules.
2. The unique maximal ideal and the residue field of a local ring R are respectively denoted by $\mathfrak{m}$ and $R/\mathfrak{m} = k$.
3. Unless otherwise specified, all Homs and $\otimes$s are over R.

Definition 1 Let $\mathbf{P}_\bullet$ and $\mathbf{Q}_\bullet$ be projective resolutions of M and N respectively over R. (It is actually enough to consider flat resolutions.) One defines

$$\mathrm{Tor}_i^R(M,N) \overset{\text{def}}{=} H_i(\mathbf{P}_\bullet \otimes N) \overset{(*)}{=} H_i(M \otimes \mathbf{Q}_\bullet)$$

i.e., Tor is the left derived functor of the right exact functors $\_ \otimes N$ and $M \otimes \_$.

Proof of $(*)$: Consider the following maps of double complexes:

$$\begin{tikzcd} \mathbf{P}_\bullet \otimes \mathbf{Q}_\bullet \arrow[r, "f"] \arrow[d, "g"] & \mathbf{P}_\bullet \otimes N \\ M \otimes \mathbf{Q}_\bullet & \end{tikzcd}$$

where $\mathbf{P}_\bullet \otimes \mathbf{Q}_\bullet$ is

$$\begin{tikzcd} {} \arrow[d] & {} \arrow[d] & \\ P_0 \otimes Q_1 \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & {} \arrow[l] \\ P_0 \otimes Q_0 & P_1 \otimes Q_0 \arrow[l] & {} \arrow[l] \end{tikzcd}$$

$\mathbf{P}_\bullet \otimes N$ is

$$\begin{tikzcd} {} \arrow[d] & {} \arrow[d] & \\ 0 \arrow[d] & 0 \arrow[l] \arrow[d] & {} \arrow[l] \\ P_0 \otimes N & P_1 \otimes N \arrow[l] & {} \arrow[l] \end{tikzcd}$$

and $M \otimes \mathbf{Q}_\bullet$ is

$$\begin{tikzcd} {} \arrow[d] & {} \arrow[d] & \\ M \otimes Q_1 \arrow[d] & 0 \arrow[l] \arrow[d] & {} \arrow[l] \\ M \otimes Q_0 & 0 \arrow[l] & {} \arrow[l] \end{tikzcd}$$

Since each term of $\mathbf{P}_\bullet$ is projective, the map f induces a quasiisomorphism of the columns of $\mathbf{P}_\bullet \otimes \mathbf{Q}_\bullet$ and $\mathbf{P}_\bullet \otimes N$ (i.e., an isomorphism of their homologies), and so the map on the corresponding total complexes is also a quasi-isomorphism. Similarly, g induces a quasiisomorphism of the total complexes of $\mathbf{P}_\bullet \otimes \mathbf{Q}_\bullet$ and $M \otimes \mathbf{Q}_\bullet$ since it does so on the rows. Hence we have

$$H_i(\mathbf{P}_\bullet \otimes N) \xleftarrow[f]{\cong} H_i(\mathbf{P}_\bullet \otimes \mathbf{Q}_\bullet) \xrightarrow[g]{\cong} H_i(M \otimes \mathbf{Q}_\bullet)$$

which proves $*$. □

Note that:

(a) Tor is a functor of either variable, and

(b) $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_i^R(N,M)$ (use $(*)$ and the fact that $\otimes$ is symmetric).

Definition 2 Let $\mathbf{P}_\bullet$ be a projective resolution of M and $\mathbf{I}^\bullet$ be an injective resolution of N over R. One defines

$$\mathrm{Ext}_R^i(M,N) \overset{\text{def}}{=} H^i(\mathrm{Hom}(\mathbf{P}_\bullet, N)) \overset{(*)}{=} H^i(\mathrm{Hom}(M, \mathbf{I}^\bullet))$$

i.e., Ext is the right derived functor of the left exact functors $\mathrm{Hom}(\_\,, N)$ and $\mathrm{Hom}(M, \_\,)$.

Long Exact Sequences of Ext and Tor

1. Consider the short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$.

(a) Applying $\_ \otimes \mathbf{Q}_\bullet$ gives a short exact sequence of complexes which induces the following long exact sequence on Tors:

$$\cdots \to \mathrm{Tor}_i^R(M_3,N) \xrightarrow{\partial} \mathrm{Tor}_{i-1}^R(M_1,N) \to \mathrm{Tor}_{i-1}^R(M_2,N) \to \mathrm{Tor}_{i-1}^R(M_3,N) \xrightarrow{\partial}$$

$$\cdots \to \mathrm{Tor}_1^R(M_3,N) \xrightarrow{\partial} M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$$

(b) Applying $\mathrm{Hom}(\_\,, \mathbf{I}^\bullet)$ gives a short exact sequence of complexes which induces the following long exact sequence on Exts:

$$0 \to \mathrm{Hom}(M_3,N) \to \mathrm{Hom}(M_2,N) \to \mathrm{Hom}(M_1,N) \xrightarrow{\partial} \mathrm{Ext}_R^1(M_3,N) \to \cdots$$

$$\xrightarrow{\partial} \mathrm{Ext}_R^i(M_3,N) \to \mathrm{Ext}_R^i(M_2,N) \to \mathrm{Ext}_R^i(M_1,N) \xrightarrow{\partial} \mathrm{Ext}_R^{i+1}(M_3,N) \to \cdots$$

2. Similarly the short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ induces long exact sequences on Tors and Exts respectively by applying (a) $\mathbf{P}_\bullet \otimes \_$ and (b) $\mathrm{Hom}(\mathbf{P}_\bullet, \_\,)$.

Some **Consequences:**
(a) $\mathrm{Tor}_i(M, \_\,) = 0$ for all $i \iff M \otimes \_\,$ is exact $\iff M$ is flat.
(b) $\mathrm{Ext}^i(M, \_\,) = 0$ for all $i \iff \mathrm{Hom}(M, \_\,)$ is exact $\iff M$ is projective.
(c) $\mathrm{Ext}^i(\_\,, N) = 0$ for all $i \iff \mathrm{Hom}(\_\,, N)$ is exact $\iff N$ is injective.

Remarks:
1. It is enough to check the vanishing of Tor and Ext for $i = 1$.
2. If M, N are finitely generated and R is local, then it is enough to check the vanishing of the Tor and Ext against the residue field k.

Applications of Tor and Ext

We give three "examples" to illustrate some uses of Tor and Ext.

1. Depth Recall that $\mathrm{depth}_I M$ is defined to be the length of a maximal M-regular sequence in I. Then

$$\mathrm{depth}_I M = \min\{i : \mathrm{Ext}^i_R(R/I, M) \neq 0\}$$

$$\text{i.e., } \mathrm{depth}_I M \geq t \iff \mathrm{Ext}^i_R(R/I, M) = 0 \text{ for all } i < t.$$

Proof: Induct on $t = \mathrm{depth}_I M$. We will use the fact that for any finitely generated R-module N one has

$$\mathrm{Hom}(N, M) = 0 \iff \mathrm{depth}_{\mathrm{ann}_R N} M > 0$$

If $t = 0$, then the above fact yields $\mathrm{Hom}(R/I, M) \neq 0$. Hence $\mathrm{depth}_I M = 0 = \min\{i : \mathrm{Ext}^i_R(R/I, M) \neq 0\}$ as desired.

Now suppose $t > 0$. Let x be a non-zerodivisor on M. Applying $\mathrm{Hom}_R(R/I, \_\,)$ to the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, yields a long exact sequence that breaks up into exact sequences

$$0 \to \mathrm{Hom}(R/I, M/xM) \to \mathrm{Ext}^1_R(R/I, M) \to 0$$

(by the fact above again) and, for $i > 0$,

$$0 \to \mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I, M/xM) \to \mathrm{Ext}^{i+1}_R(R/I, M) \to 0$$

By induction, for $i < t-1$, $\mathrm{Ext}^i_R(R/I, M/xM) = 0$ and $\mathrm{Ext}^{t-1}_R(R/I, M/xM) \neq 0$. This forces $\mathrm{Ext}^i_R(R/I, M) = 0$ for $i < t$ and $\mathrm{Ext}^t_R(R/I, M) \neq 0$, proving the result. $\square$

2. Betti numbers Let $(R, \mathfrak{m}, k)$ be a local ring. Recall that a free resolution of an R-module M

$$\mathbf{F}_\bullet : \quad \ldots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \ldots \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} 0$$

is said to be *minimal* if $\mathrm{im}(\partial_i) \subseteq \mathfrak{m}F_{i-1}$ for each i. The ith *Betti number* of M is defined to be $b_i(M) \stackrel{\text{def}}{=} \mathrm{rank}(F_i)$. Then

$$b_i(M) = \dim_k \mathrm{Tor}_i^R(M,k)$$

Proof: Since $\partial_i \otimes k = 0$ for each $i > 0$, we have

$$\mathrm{Tor}_i^R(M,k) = H_i(\mathbf{F}_\bullet \otimes k) = F_i \otimes k$$

But $F_i \otimes k \cong k^{\mathrm{rank} F_i}$, which proves the required equality. $\square$

Note that for any M, $\mathrm{Tor}_i^R(M,k) = 0$ forces $b_j(M) = 0$ for $j > i$. This gives us $\mathrm{Tor}_j^R(M,k) = 0$ for $j > i$. We say that k is *rigid*.

3. Koszul (co)homology Suppose that $\underline{x} = x_1, \ldots, x_n$ is an R-regular sequence. In this case, the Koszul complex $\mathbf{K}_\bullet$ is a free resolution of $R/(\underline{x})$, and so the Koszul homologies (and cohomologies) are:

$$H_i(\underline{x}, M) = H_i(\mathbf{K}_\bullet \otimes M) = \mathrm{Tor}_i^R(R/\underline{x}, M)$$

$$H^i(\underline{x}, M) = H_i(\mathrm{Hom}(\mathbf{K}_\bullet, M)) = \mathrm{Ext}_R^i(R/\underline{x}, M)$$

Change of Rings

Let S be a ring, M and N be S-modules. Let $x \in S$ be an S-regular and M-regular element such that $xN = 0$. Let $\overline{M} = M/xM$ and $R = S/xS$. Then

1. $\mathrm{Ext}_S^n(M,N) \cong \mathrm{Ext}_R^n(\overline{M},N)$
2. $\mathrm{Tor}_n^S(M,N) \cong \mathrm{Tor}_n^R(\overline{M},N)$
3. $\mathrm{Ext}_S^{n+1}(N,M) \cong \mathrm{Ext}_R^n(N,\overline{M})$

Idea of Proof: The first two isomorphisms follow from the fact that if $\mathbf{F}_\bullet$ is a free resolution of M over S, then $\mathbf{F}_\bullet \otimes_S R$ is a free resolution of $\overline{M} \cong M \otimes_S R$ over R. (To see this, resolve R using the Koszul complex $0 \to S \xrightarrow{x} S \to R \to 0$ and thus compute that $\mathrm{Tor}_i^S(M,R) = H_i(\mathbf{F}_\bullet \otimes_S R) = 0$ for all $i > 0$.) The third isomorphism follows from the following more general result which can be proven using a spectral sequence argument.

Proposition 1 *Let S be a ring, and let M and N be S-modules. Let x be an S-regular element with xM=0. Set $R = S/xS$. Then there is a long exact sequence*

$$\cdots \to \mathrm{Ext}_R^i(M,(0:x)_N) \to \mathrm{Ext}_S^i(M,N) \to \mathrm{Ext}_R^{i-1}(M,\overline{N}) \to$$
$$\to \mathrm{Ext}_R^{i+1}(M,(0:x)_N) \to \mathrm{Ext}_S^{i+1}(M,N) \to \mathrm{Ext}_R^i(M,\overline{N}) \to \cdots .$$

Corollary 2 *Let S be a ring and x an S-regular element, and set $R = S/xS$. For any R-modules M and N, there is a long exact sequence*

$$\begin{aligned} \cdots \to \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_S(M,N) \to \operatorname{Ext}^{i-1}_R(M,N) \to \\ \to \operatorname{Ext}^{i+1}_R(M,N) \to \operatorname{Ext}^{i+1}_S(M,N) \to \operatorname{Ext}^i_R(M,N) \to \cdots \end{aligned}$$

§ 2 *The Frobenius Endomorphism*

Introduction

In this section, we assume that the (Noetherian for simplicity) ring R has positive characteristic p. In such a case we have the Frobenius endomorphism $f\colon R \to R$ defined by $r \mapsto r^p$ and its compositions $f^n : R \to R$ with $f^n(r) = r^{p^n}$.

Note that this is a ring map since for any $r,\ s \in R$, $(r+s)^p = r^p + s^p$ in positive characteristic p, the other binomial coefficients being divisible by p.

If I is an ideal in R, the extension ideal $f^n(I)$ is denoted by $I^{[p^n]}$. If $I = (a_1, \ldots, a_t)$, then $I^{[p^n]} = (a_1^{p^n}, \ldots, a_t^{p^n})$, the ideal generated by the pure powers of the generators of I.

Observe that $I^{[p^n]} \subseteq I^{p^n}$. Usually I^{p^n} is much larger than $I^{[p^n]}$; however they are the same up to radical.

Two Functors

Restriction of scalars along f^n:

Let M be an R-module. We write ${}^{f^n}R$ and ${}^{f^n}M$ for the left R-module structure defined on R and M, respectively, by restriction of scalars via f^n, that is,

$$\text{for } r, s \in {}^{f^n}R, m \in {}^{f^n}M, \text{ one has } r \cdot s = r^{p^n}s \text{ and } r \cdot m = r^{p^n}m$$

The functor $M \mapsto {}^{f^n}M$ is exact since exactness is checked on the underlying abelian group structure (as abelian groups, M and ${}^{f^n}M$ are the same, and homomorphisms are unchanged under the functor).
Extension of scalars (base change) along f^n:

Let M be an R-module. The "Frobenius functor" (introduced by Peskine and Szpiro) from the category of R-modules to itself is given by base change along f:

$$F_R(M) = M \otimes_R {}^{f}R$$

with an R-module structure via the usual multiplication on the righthand factor, that is,

$$\text{for } r, s \in R, m \in M, (m \otimes s)r = m \otimes rs \text{ but } rm \otimes s = m \otimes r^p s.$$

Note: One can check that the compositions of the Frobenius functor are given by base change along the compositions f^n of f $F_R^n(M) \cong M \otimes_R {}^{f^n}R$.

Remark: To make the Frobenius functor easier to understand, one can write the (iterated) Frobenius endomorphism as $R \xrightarrow{f^n} S$, where $S = R$, in order to distinguish between the two domain and target rings R. As an R-module $S = {}^{f^n}R$:

$$\text{for } r \in R, s \in S(= R), \text{ one has } r \cdot s = f^n(r)s = r^{p^n}s$$

Similarly we can think of ${}^{f^n}M$ as the R-module structure on the S-module M by restriction of scalars. With this notation, it is now clearer that

$$F_R^n(M) = M \otimes_R S$$

with its natural $S(=R)$-module structure being via the usual multiplication on the righthand factor.

To illustrate how this notation can help, we justify the note above in the case of $n = 2$. Consider $R = S = T$ with maps $R \xrightarrow{f} S \xrightarrow{f} T$. By the remark above, the R-module structures on S and T are the same as those of ${}^fR$ and ${}^{f^2}R$, respectively. Thus

$$F_R^2(M) = F_S(M \otimes_R S) = (M \otimes_R S) \otimes_S T \cong M \otimes_R T = M \otimes_R {}^{f^2}R$$

Properties of the Frobenius Functor

We begin with a list of the basic properties of the Frobenius functor each followed by a brief justification using the alternate notation introduced in the remark above, namely $R \xrightarrow{f^n} S$, where $S = R$, for the nth iteration of the Frobenius endomorphism.

1) The functor F is right exact (since tensor product is right exact).

2) $F_R^n(R) = R \otimes_R {}^{f^n}R \cong R$ as R-modules. This is easier to see in the alternate notation: $F_R^n(R) = R \otimes_R S \cong S$ as S-modules.

3) $F_R^n(R^t) \cong R^t$. (This follows from (2) since the tensor product commutes with finite direct sums).

4) $F_R^n(R/I) = R/I \otimes_R {}^{f^n}R \cong R/f^n(I)R = R/I^{[p^n]}R$ for an ideal I in R. To see this, we compute in the alternate notation: As S-modules, $F_R^n(R/I) = R/I \otimes_R S \cong S/f^n(I)S$.

5) Let r be an element of R. Consider the map $R \xrightarrow{r} R$ given by multiplication by r. Then $F_R^n(R \xrightarrow{r} R) = R \xrightarrow{r^{p^n}} R$, that is, multiplication by r^{p^n}.
Indeed, again using the alternate notation, we see that

$$F^n(R \xrightarrow{\cdot r} R) = (R \otimes_R S \xrightarrow{r \otimes 1} R \otimes_R S) = (S \xrightarrow{f^n(r)} S)$$

5′) As a consequence of (5) and (3), we can describe the effect of F^n on a map between finitely generated free modules, say given by and $s \times t$ matrix $[r_{ij}]$ with $r_{ij} \in R$:

$$F^n(R^t \xrightarrow{[r_{ij}]} R^s) = R^t \xrightarrow{[r_{ij}^{p^n}]} R^s$$

5″) In particular, this gives an explicit description of $F^n(M)$: if

$$R^t \xrightarrow{[r_{ij}]} R^s \to M \to 0$$

is a presentation of an R-module M, then applying F^n yields a presentation

$$R^t \xrightarrow{[r_{ij}^{p^n}]} R^s \to F^n(M) \to 0$$

of $F^n(M)$ since F^n is right exact.

Note that this gives another proof of (4). For an ideal $I = (r_1, \dots, r_t)$,

$$R^t \xrightarrow{[r_1,\dots,r_t]} R \to R/I \to 0$$

is a presentation of R/I. So, a presentation of $F^n(R/I)$ is

$$R^t \xrightarrow{[r_1^{p^n},\dots,r_t^{p^n}]} R \to F^n(R/I) \to 0$$

5‴) From (5) we see that if M is a finitely generated R-module, then so is $F^n(M)$.

6) For any prime $\mathfrak{p}$ in Spec$R$,

$$F^n_R(M)_{\mathfrak{p}} \cong F^n_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

To see this, first note that

$$(f^n)^{-1}(\mathfrak{p}) = \{r \in R : f^n(r) = r^{p^n} \in \mathfrak{p}\} = \mathfrak{p}$$

Hence, in the alternate notation, we have

$$F^n_R(M)_{\mathfrak{p}} = (M \otimes_R S)_{\mathfrak{p}} \cong M \otimes_R S_{\mathfrak{p}} \cong M_{(f^n)^{-1}(\mathfrak{p})} \otimes_{R_{(f^n)^{-1}(\mathfrak{p})}} S_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{p}} = F^n_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

7) Let M be a finitely generated R-module. If M is nonzero, then so is $F^n_R(M)$. Indeed, using a prime filtration of M one can obtain a surjective map $M \twoheadrightarrow R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$. Since F^n_R is right exact, we get a surjection $F^n_R(M) \twoheadrightarrow F^n_R(R/\mathfrak{p}) \cong R/\mathfrak{p}^{[p^n]} \neq 0$. Therefore, $F^n_R(M)$ is nonzero.

8) As an immediate consequence of (6) and (7), we see that Supp$F^nR(M) =$ Supp$M$ for a finitely generated R-module M.

9) Even if M is finitely generated, the associated primes of $F^n(M)$ are not predictable in general. On a related note, there are even examples of modules over local rings for which the depth of $F^n(M)$ can be higher or lower than that of M itself.

However, if $\mathrm{pd}_R M < \infty$, then in fact $\mathrm{Ass}_R F^n(M) = \mathrm{Ass}_R M$, as we will see later (refer Cor. 3).

Left Derived Functors of F^n: Since the functor F^n is simply the tensor product with ${}^{f^n}R$, its left derived functors are given by the Tors: If $\mathbf{P}_\bullet$ is a projective resolution of M over R, then

$$H_i(F^n(\mathbf{P}_\bullet)) = H_i(\mathbf{P}_\bullet \otimes_R {}^{f^n}R) = \mathrm{Tor}_i^R(M, {}^{f^n}R)$$

This homology group inherits its R-module structure from the structure on $P_i \otimes_R {}^{f^n}R$ via multiplication on the righthand factor. The R-module $\mathrm{Tor}_i^R(M, {}^{f^n}R)$ is finitely generated if M is, localises well and satisfies $\mathrm{Supp}(\mathrm{Tor}_i^R(M, {}^{f^n}R)) \subseteq \mathrm{Supp}M$.

Roles of the Frobenius Endomorphism in Commutative Algebra

The Frobenius endomorphism plays two, initially somewhat intertwined, roles in commutative algebra. Its central place in the investigation of problems in the field was introduced and cemented by the fundamental papers of Kunz in 1969 and of Peskine and Szpiro a few years later.

Its main role is as a tool for proving results in positive characteristic p, as the Frobenius can be used to "twist" a given situation and until it provides the desired conclusion or contradiction. This method was introduced by Peskine and Szpiro to prove the Intersection Theorem in characteristic p. We will not discuss this aspect of the Frobenius although it is behind the proof of many of the homological conjectures that will be discussed in the second week. However, similar ideas come up in some of the proofs below.

In the same paper Peskine and Szpiro introduced the method of reduction to characteristic p to obtain their result for (many) rings of equicharacteristic zero from their result in characteristic p. Hochster refined this technique to work for all such rings; we give a brief discussion at the end of the section.

Another role that the Frobenius plays is in providing a test module: ${}^{f^n}R$ can be used to detect certain properties of R or an R-module M in the same way that the residue field k does. As the discussion of this aspect both involves the early history mentioned above and provides some useful properties of the Frobenius endomorphism and functor, we spend most of the section on this topic.

We begin with Kunz's surprising and fundamental result. Kunz was initially interested in obtaining new numerical invariants (from condition (3) below) other than the usual multiplicity for studying resolution of singularities in characteristic p, and indeed the study of Hilbert-Kunz multiplicities (which has close relations to tight closure - see the talks by F. Enescu) grew from this. However, Kunz's Theorem produced an unexpected direction (via condition (2)) of research, which we discuss further below.

Theorem 1 (Kunz) *Let R be a local ring of characteristic p and dimension d. The following are equivalent:*
1. *R is regular. (i.e.,* $\mathrm{flatdim}_R k < \infty$*).*
2. *f^n is flat for some (all) $n > 0$. (i.e.,* $\mathrm{flatdim}_R {}^{f^n}R = 0$*).*
3. *$\ell(F_n^R(k)) = p^{nd}$ for some (all) $n > 0$.*

Partial Proof of (1) $\Leftrightarrow$ (2): For (1) $\Rightarrow$ (2), we may assume that R is complete since $R \hookrightarrow \widehat{R}$ is faithfully flat. Then Cohen's Structure Theorem implies that $R \cong k[[X_1, \ldots, X_d]]$ (where k is a coefficient field of R).

We have

$$\begin{array}{ccc} k[[X_1, \ldots, X_d]] & \xrightarrow{f^n} & k[[X_1, \ldots, X_d]] \\ \downarrow \cong & & \uparrow \mathit{free} \\ k^{p^n}[[X_1^{p^n}, \ldots, X_d^{p^n}]] & \longrightarrow & k[[X_1^{p^n}, \ldots, X_d^{p^n}]] \end{array}$$

It is enough to show that the map $A \stackrel{\mathrm{def}}{=} k^{p^n}[[X_1^{p^n}, \ldots, X_d^{p^n}]] \to B \stackrel{\mathrm{def}}{=} k[[X_1^{p^n}, \ldots, X_d^{p^n}]]$ is flat. One can use the local criterion for flatness to prove this: To prove that B is flat over A, it is enough to show that $\mathrm{Tor}_i^A(k_A, B) = 0$ for all $i > 0$, which is a straightforward computation using the Koszul complex to resolve k_A over A.

For the proof (2) $\Rightarrow$ (1) Kunz begins by proving that (2) implies the condition

$$(*) \quad \ell(F^n(k)) = p^{ne}, \text{ where } e = \mu(\mathfrak{m}), \text{ the embedding dimension of } R$$

using results of Lech. We assume this part and resume with the rest of the proof. Since f^n is flat, its compositions with itself are flat. Hence we may assume that f^n is flat for infinitely many values of n and so that condition (2), and hence condition $(*)$, actually hold for infinitely many values of n. Furthermore, we may assume that R is complete. Hence by Cohen's Structure Theorem, $R \cong k[[X_1, \ldots, X_e]]/I$ for some ideal I. Note that $\mathfrak{m} = (\overline{X}_1, \ldots, \overline{X}_e)$ and so that $\mathfrak{m}^{[p^n]} = (\overline{X}_1^{p^n}, \ldots, \overline{X}_e^{p^n})$.

Note that

$$F^n(k) = R/\mathfrak{m}^{[p^n]} \cong k[[X_1, \ldots, X_e]]/(I + \mathfrak{m}^{[p^n]})$$

So, $(*)$ yields

$$\dim_k k[[X_1, \ldots, X_e]]/(I + \mathfrak{m}^{[p^n]}) = p^{ne}$$

for infinitely many values of n. On the other hand, by counting an obvious basis, one obtains

$$\dim_k k[[X_1, \ldots, X_e]]/\mathfrak{m}^{[p^n]} = p^{ne}$$

as well. Since one is a quotient of the other and both have the same k-dimensions,

$$k[[X_1, \ldots, X_e]]/\mathfrak{m}^{[p^n]} = k[[X_1, \ldots, X_e]]/(I + \mathfrak{m}^{[p^n]}),$$

i.e., $I \subseteq \mathfrak{m}^{p^n}$, for infinitely many values of n. Hence Krull Intersection Theorem forces I to be zero. Thus $R \cong k[[X_1, \ldots, X_e]]$ and hence is regular. $\square$

The translation in parentheses of condition (1) in Kunz's Theorem is the celebrated Auslander-Buchsbaum-Serre Theorem; it shows how the residue field k functions as a test module for the regularity of R. The (immediate) translation in parentheses of condition (2) in Kunz's Theorem make it clear that the R-module ${}^{f^n}R$ performs the same role.

With the next theorem, a generalization to modules, the parallel between k and ${}^{f^n}R$ as test modules is continued since $\mathrm{pd}_R M < \infty$ if and only if $\mathrm{Tor}_i^R(M, k) = 0$ for all $i > 0$. Note also that, applying the result to *all* R-modules M at once, one retrieves Kunz's Theorem.

Theorem 2 (Peskine-Szpiro; Herzog) *Let R be a local ring of characteristic p. For any finitely generated R-module M, the following are equivalent:*
1. $\mathrm{pd}_R M < \infty$.
2. $\mathrm{Tor}_i^R(M, {}^{f^n}R) = 0$ *for all* $i > 0$ *and all (infinitely many)* $n > 0$.

Peskine and Szpiro proved that (1) implies (2) to use as a major ingredient in their proof of the Intersection theorem. The converse was proved by Herzog.

Corollary 3 *Let R be a regular local ring. If M is an R-module with (minimal) free resolution $\mathbf{F}_\bullet$, then $F^n(\mathbf{F}_\bullet)$ is a (minimal) free resolution of $F^n(M)$.*

Remark: The corollary implies that if $\mathrm{pd}_R M < \infty$,

$$\mathrm{pd}_R M = \mathrm{pd}_R F^n(M)$$

Hence, in case R is local, the Auslander-Buchsbaum Formula yields further that

$$\mathrm{depth} F^n(M) = \mathrm{depth} M$$

In particular, $\mathfrak{m} \in \mathrm{Ass}_R(F^n(M)) \iff \mathfrak{m} \in \mathrm{Ass}_R(M)$. By localising and using the previous argument, we see that if $\mathrm{pd}_R M < \infty$,

$$\mathrm{Ass}_R(F^n(M)) = \mathrm{Ass}_R(M)$$

Definition 1 If $\mathbf{L}_\bullet$ is a complex such that $H_i(\mathbf{L}_\bullet) = 0$ for all $i > 0$, then we say that $\mathbf{L}_\bullet$ is acyclic.

Note that a complex $\mathbf{L}_\bullet$ of projective modules is acyclic if and only if it is a resolution of $H_0(\mathbf{L}_\bullet)$.

Sketch of Proof of (1) $\Rightarrow$ (2) in Theorem 2: Note that $H_i(F^n(\mathbf{F}_\bullet)) = \mathrm{Tor}_i^R(M, {}^{f^n}R)$. Suppose that for some $i > 0$ $\mathrm{Tor}_i^R(M, {}^{f^n}R) \neq 0$. Choose a minimal prime $\mathfrak{p}$ in $\bigcup_{i>0} \mathrm{Supp}(\mathrm{Tor}_i^R(M, {}^{f^n}R))$ (this set is not empty). Then for each $i > 0$ we have that $\ell(\mathrm{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, {}^{f^n}R_\mathfrak{p})) = \ell((\mathrm{Tor}_i^R(M, {}^{f^n}R))_\mathfrak{p}) < \infty$. So, replacing R by $R_\mathfrak{p}$ and M by $M_\mathfrak{p}$, we may assume that $(R, \mathfrak{m}, k)$ is local and that $\ell(\mathrm{Tor}_i^R(M, {}^{f^n}R)) < \infty$ for each $i > 0$ and at least one is nonzero.

Let $\mathbf{F}_\bullet$ be a minimal free resolution of M over R of length $s = \mathrm{pd}_R M$. By the Auslander-Buchsbaum formula, $s \leq \mathrm{depth} R$. So, the complex $\mathbf{L}_\bullet = F^n(\mathbf{F}_\bullet) = \mathbf{F}_\bullet \otimes {}^{f^n}R$ satisfies $\ell(H_i(\mathbf{L}_\bullet)) < \infty$ for all $i > 0$ and has length less than $\mathrm{depth} R$, but is *not* acyclic. This contradicts the following result which is a special case of the Acyclicity Lemma proved by Peskine and Szpiro which states:

Lemma 4 (A version of the Acyclicity Lemma) *Let $0 \to L_s \to L_{s-1} \to \cdots \to L_0 \to 0$ be a complex of free finitely generated R-modules with $s \leq \mathrm{depth} R$. If $\ell(H_i(\mathbf{L}_\bullet)) < \infty$ for all $i > 0$, then $H_i(\mathbf{L}_\bullet) = 0$ for all $i > 0$.*

In the more general version of the Acyclicity Lemma given by Peskine and Szpiro it is enough to assume that the L_i are finitely generated R-modules such that $\mathrm{depth} L_i \geq i$ and for each $i > 0$ either $\mathrm{depth} H_i(\mathbf{L}_\bullet) = 0$ or $H_i(\mathbf{L}_\bullet) = 0$.

Before continuing with the Frobenius, we review some other acyclicity lemmas. The most complete answer is given by the following result. We first set some notation. Let

$$\mathbf{L}_\bullet : \quad 0 \longrightarrow L_s \xrightarrow{\partial_s} L_{s-1} \xrightarrow{\partial_{s-1}} \cdots \xrightarrow{\partial_1} L_0 \to 0$$

be a complex of finitely generated free R-modules. Set $r_i = \Sigma_{j=i}^s (-1)^{j-i} \mathrm{rank} L_j$; this is the expected rank of $\mathrm{im}(\partial_i)$ if $\mathbf{L}_\bullet$ were acyclic. Let $I_{r_i}(\partial_i)$ is the ideal of $r_i \times r_i$ minors of the matrix of ∂_i.

Theorem 5 (Buchsbaum-Eisenbud Acyclicity Criterion) *With the notation above, we have that*

$$\mathbf{L}_\bullet \text{ is acyclic} \iff \mathrm{depth}_{I_{r_i}(\partial_i)} R \geq i \text{ for all } i > 0$$

The following lemma is used in proving the acyclicity criterion, and will be used in the example of reduction to characteristic p given at the end of this section.
A complex $\mathbf{L}_\bullet$ is said to be *split acyclic* if it is acyclic and $(\mathrm{im}\partial_i)_\mathfrak{p}$ is a direct summand of L_{i-1}, that is, if the complex splits all the way along from the left except that the last differential may not be surjective.

Lemma 6 *With notation as before, for a prime ideal $\mathfrak{p}$ in R,*

$$(\mathbf{L}_\bullet)_\mathfrak{p} \text{ is split acyclic} \iff I_{r_i}(\partial_i) \not\subseteq \mathfrak{p} \text{ for all } i > 0$$

Reduction to Characteristic p

Since the Frobenius endomorphism is an effective tool for proving theorems in positive characteristic, it becomes important to have a way of reducing results in characteristic zero case to this case. The first such method in commutative algebra was introduced by Peskine and Szpiro in [4]. They proved the Intersection Theorem, Auslander's Conjecture and Bass's Conjecture in characteristic p and then used Artin Approximation Theory to deduce these results for rings that are essentially finite over fields of characteristic zero from the positive characteristic case. Hochster then refined the technique to apply to all rings of equicharacteristic zero, and this is what we present here.

The exposition in this section and the example below are directly from [1], but are included here for completeness as they have shown to be an important method for establishing the characteristic zero cases of many of the homological conjectures.

The key idea is to be able to describe the existence of a counterexample to a statement one wishes to prove in terms of polynomial equations over the integers $\mathbb{Z}$, and then to apply the result below.

Definition 2 A subset $\mathfrak{E} \subseteq \mathbb{Z}[\underline{X}, \underline{Y}]$ is called a *system of equations over* $\mathbb{Z}$. We say that $\mathfrak{E}$ *has a solution of height n in R* if there exist $\underline{x} = x_1, \ldots, x_n$ and $\underline{y} = y_1, \ldots, y_m$ in R such that
(i) $p(\underline{x}, \underline{y}) = 0$ for all $p \in \mathfrak{E}$, and
(ii) $\mathrm{ht}(\underline{x}) = n$.

Theorem 7 (Hochster) *Let $\mathfrak{E}$ be a system of equations over $\mathbb{Z}$. If $\mathfrak{E}$ has a solution $\underline{x} = x_1, \ldots, x_n$ and $\underline{y} = y_1, \ldots, y_m$ of height n in a Noetherian ring R containing a field, then*

a) *$\mathfrak{E}$ has a solution $\underline{x}', \underline{y}'$ of height n in an affine domain R' containing a finite field and*

b) *$\mathfrak{E}$ has a solution $\underline{x}', \underline{y}'$ in a local ring R' containing a field of positive characteristic p such that $\underline{x}'$ is a system of parameters for R'.*

Note that (b) is obtained by localising (a) at a minimal prime over $\underline{x}'$.

Strategy:

1. Prove the result in characteristic p. (By (b) it suffices to prove it for local rings.)

2. Show that there is a family $(\mathfrak{E}_i)_{i\in I}$ of systems of equations over $\mathbb{Z}$ such that, for any ring R, the statement holds for R if and only if none of the systems $\mathfrak{E}_i$ has a solution of the appropriate height in R. In other words, one wants that

$$\begin{gathered}\text{There is a counterexample over } R\\ \Updownarrow\\ \text{One of the systems } \mathfrak{E}_i \text{ has a solution of the appropriate height in } R\end{gathered}$$

3. The theorem then applies to give the result in equicharacteristic zero.

Hochster used this strategy to show that every Noetherian local ring R of equal characteristic (i.e., containing a field) has a big Cohen-Macaulay module and used that result to solve some homological conjectures.

An Example of Reduction to Characteristic p

We now give an example, taken from [1], of the reduction (Step 2 above), namely of the reduction of the New Intersection Theorem to characteristic p. This theorem was proved independently by Peskine and Szpiro and by Roberts in the equal characteristic case. Later Roberts proved it in the mixed case as well.

Theorem 8 (The New Intersection Theorem) *Consider a complex*

$$\mathbf{F}_\bullet\colon \quad 0 \longrightarrow F_s \xrightarrow{\partial_s} F_{s-1} \xrightarrow{\partial_{s-1}} \cdots \xrightarrow{\partial_1} F_0 \to 0$$

of finitely generated free R-modules with the property that $\ell(H_i(\mathbf{F}_\bullet)) < \infty$ for each i. If $s < \dim R$, then $\mathbf{F}_\bullet$ is (split) exact.

First note that may assume that R is local. For the reduction to the positive characteristic p case, we need to express the existence of a counterexample over a local ring R as the existence of a solution $\underline{x}, \underline{y}$ in R (where $\underline{x}$ is a system of parameters for R) of one of a family of systems $\mathfrak{E}_i$ of polynomial equations over $\mathbb{Z}$.

With this intent, we note that a counterexample would consist of maps

$$\mathbf{F}_\bullet : \quad 0 \to F_s \xrightarrow{[y_{ij}^{(s)}]} F_{s-1} \longrightarrow \cdots \xrightarrow{[y^{(1)}]_{ij}} F_0 \to 0$$

of free R-modules with $s < \dim R$ that satisfy the following conditions:

a) To ensure that $\mathbf{F}_\bullet$ is a complex, one requires the products of consecutive differentials to be zero:

$$[y_{ij}^{(k)}][y_{ij}^{(k+1)}] = 0 \qquad (1)$$

Setting the entries in the product of the matrices equal to zero yields polynomial equations in the variables $y_{ij}^{(k)}$.

b) Since a counterexample is not split exact, one may assume (splitting off F_0 if necessary and renumbering, and repeating as needed) that F_0 does not split off at the righthand end and so that $H_0(\mathbf{F}_\bullet) \neq 0$.

This is equivalent to requiring that $\mathrm{im}(\partial_1) \subseteq \mathfrak{m}F_0$, or equivalently that each $y_{ij}^{(1)} \in \mathfrak{m}$. Since $\underline{x}$ is an s.o.p. for R, this is equivalent to requiring that for some $t \in \mathbb{N}$ one has $(y_{ij}^{(1)})^t \in (\underline{x})$ for all i and j, that is, that

$$(y_{ij}^{(1)})^t = \Sigma_k a_{ijk} x_k \text{ for some } a_{ijk} \in R. \qquad (2)$$

c) Next we seek equations that ensure that $\ell(H_i(\mathbf{F}_\bullet)) < \infty$ for all i, in other words, that $(\mathbf{F}_\bullet)_\mathfrak{p}$ is exact (and thus split) for all $\mathfrak{p} \neq \mathfrak{m}$. We break this condition up into the two conditions

$$\begin{aligned} &\text{(i) } (\mathbf{F}_\bullet)_\mathfrak{p} \text{ is split acyclic for all } \mathfrak{p} \neq \mathfrak{m}, \text{ and} \\ &\text{(ii) } \Sigma(-1)^i \mathrm{rank} F_i = 0 \end{aligned}$$

By Lemma 6, condition (i) holds if and only if $I_{r_k}([y_{ij}^{(k)}]) \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \neq \mathfrak{m}$, or equivalently, $I_{r_k}([y_{ij}^{(k)}])$ is $\mathfrak{m}$-primary. Since $\underline{x}$ is an s.o.p. for R, this is equivalent to requiring that for some $u \in \mathbb{N}$ one has $(\underline{x})^u \in I_{r_k}([y_{ij}^{(k)}])$ for all k. This is equivalent to writing each monomial $m_h = x_1^{i_{1h}} \cdots x_n^{i_{nh}}$ in the x's of degree u as an R-linear combination of the minors of $[y_{ij}^{(k)}]$, that is,

$$x_1^{i_{1h}} \cdots x_n^{i_{nh}} = \Sigma_l b_{hl}^{(k)} M_l^{(k)} \qquad (3)$$

where $b_{hl}^{(k)} \in R$ and $M_l^{(k)}$ are the $r_k \times r_k$ minors of $[y_{ij}^{(k)}]$.

Condition (ii), on the other hand, is simply a restriction on the ranks of the F_i which determines F_0, thus restricting the dimensions of the system $\mathfrak{E}_i$.

Conditions (1), (2), and (3) are polynomial conditions on $\underline{x}$, y_{ij}, a_{ijk} and $b_{hl}^{(k)}$. Thus they can be expressed as the existence of solutions to a system $\mathfrak{E}_i$ of polynomial equations over $\mathbb{Z}$ on variables $\underline{X}$ representing the $\underline{x}$ and variables $\underline{Y}$ representing the y_{ij}, a_{ijk} and $b_{hl}^{(k)}$. The equations in $\mathfrak{E}_i$ depend on t, u and rank F_i, $i = 0, \ldots, s$.

So, Hochster's Theorem of reduction to positive characteristic p applies to this problem. Hence it is enough to solve it in the positive characteristic case.

§ 3 *Smoothness*

Module of (Kähler) Differentials

Definition 1: Let $A \to B$ be a homomorphism of rings. Consider the homomorphism $B \otimes_A B \xrightarrow{\mu} B$ given by $b \otimes b' \mapsto bb'$. Then

$$\begin{aligned} I \overset{\text{def}}{=} \ker(\mu) &= \text{ the ideal generated by } b \otimes 1 - 1 \otimes b, b \in B \\ &= \text{ the ideal of the diagonal } \Delta \text{ in } Y \times_X Y \end{aligned}$$

where $Y = \mathrm{Spec} B$ and $X = \mathrm{Spec} A$. The module of (Kähler) differentials of B over A is defined as

$$\Omega_{B/A} \overset{\text{def}}{=} I/I^2$$

In geometric terms, since $I/I^2 \cong I \otimes_R R/I$ where $R = B \otimes_A B$ and I is the ideal of the diagonal, the sheaf on Δ given by the module $\Omega_{B/A}$ is the restriction of the ideal sheaf of the diagonal back to the diagonal, that is, letting $i \colon \Delta \to Y \times_X Y$, we have

$$\widetilde{\Omega}_{B/A} = i^*(\widetilde{I})$$

Remarks:
(a) $\Omega_{B/A}$ is a B-module via the left factor.

(b) The map $B \xrightarrow{d} \Omega_{B/A}$ defined by $b \mapsto b \otimes 1 - 1 \otimes b$ is a homomorphism of B-modules. We shall use the symbol db to stand for $b \otimes 1 - 1 \otimes b$ from now on.

(c) By definition, $\Omega_{B/A}$ is generated by the set $\{db \mid b \in B\}$.

Example 1: $\Omega_{R[X_1,\ldots,X_n]/R}$ is a free $R[X_1,\ldots,X_n]$-module with basis $dX_1,\ldots,dX_n$. Furthermore, one can show that, for any polynomial $f \in R[X_1,\ldots,X_n]$, one has $\bar{d}(f) = \Sigma_j \frac{\partial f}{\partial X_j} dX_j$.

The First Fundamental Sequence: Consider three rings A, B and C with homomorphisms $A \xrightarrow{u} B \xrightarrow{v} C$. There is an exact sequence

$$\Omega_{B/A} \otimes_B C \xrightarrow{\tilde{v}} \Omega_{C/A} \xrightarrow{\pi} \Omega_{C/B} \to 0$$

of C-modules, where $\tilde{v}(db \otimes c) = cd(v(b))$ and $\pi(dc) = dc$.

Moreover $\tilde{v}$ is injective and split if and only if every A-derivation $B \to T$ extends to an A-derivation $C \to T$ for a C-module T.

The Second Fundamental Sequence: Consider three rings A, B and $C = B/J$ with homomorphisms $A \to B \to C$. There is an exact sequence

$$J/J^2 \xrightarrow{\bar{d}} \Omega_{B/A} \otimes_B C \xrightarrow{\tilde{v}} \Omega_{C/A} \to 0$$

of C-modules, where $\bar{d}(\bar{b}) = db \otimes 1$.

Moreover $\bar{d}$ is injective and split if and only if the the map $B/J^2 \twoheadrightarrow B/J$ splits as a map of A-algebras.

Note that since $B/J \otimes_B B/J \xrightarrow{\mu} B/J$ is an isomorphism, $\ker(\mu) = 0$ and so $\Omega_{C/B} = 0$. Thus we see that the 2nd fundamental sequence is an extension to the left of the 1st fundamental exact sequence in the case $C = B/J$.

Relation to the Geometers' Definition: We apply the 2nd fundamental sequence in order to compute $\Omega_{S/R}$ for any finitely generated R-algebra S. Write S as $S \cong R[X_1, \ldots, X_n]/J$ for some ideal J. Let $R[\underline{X}]$ denote $R[X_1, \ldots, X_n$. The second fundamental sequence gives

$$J/J^2 \xrightarrow{\bar{d}} \Omega_{R[\underline{X}]/R} \otimes_{R[\underline{X}]} S \to \Omega_{S/R} \to 0$$

Since $\Omega_{R[\underline{X}]/R}$ is a free $R[\underline{X}]$-module with basis $dX_1, \ldots, dX_n$ (see Example 1), the S-module $\Omega_{R[\underline{X}]/R} \otimes_{R[\underline{X}]} S$ is free with basis $dX_1 \otimes 1, \ldots, dX_n \otimes 1$ which we denote again by $dX_1, \ldots, dX_n$. If $J = (f_1, \ldots, f_g)$, then together with Example 1 this yields

$$\Omega_{S/R} \cong \frac{\bigoplus SdX_i}{\langle \bar{d}(f_i) \rangle} = \frac{\bigoplus SdX_i}{\langle \Sigma_j \frac{\partial f_i}{\partial X_j} dX_j \rangle} = \operatorname{coker}(S^g \xrightarrow{\left[\frac{\partial f_i}{\partial X_j}\right]} S^n)$$

Smoothness

Note that the notion of smoothness considered here is equivalent to the notion of 0-smoothness in [2], that is, smoothness with respect to the discrete topology.

Definition 2: We say that a ring homomorphism $R \to S$ is *formally smooth* if for every R-algebra T, with an ideal $J \subseteq T$, we have that

$$\mathrm{Hom}_{R-alg}(S, T/J^2) \to \mathrm{Hom}_{R-alg}(S, T/J)$$

is surjective, that is, any diagram of the form

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ T/J^2 \arrow[r] & T/J \end{tikzcd}$$

has a lift $S \to T/J^2$. One can think of this as saying that any map to T/J can be extended to an infintesimal neighborhood of J. If, furthermore, S is essentially finite over R (i.e., S is a localisation of a finitely generated R-algebra), we say that S is *smooth* over R.

Remark: Replacing T by T/J^2, in order to verify that $R \to S$ is smooth (or formally smooth), it is enough to check that for every R-algebra T with an ideal J such that $J^2 = 0$, the map $\mathrm{Hom}_{R-alg}(S,T) \to \mathrm{Hom}_{R-alg}(S,T/J)$ is surjective. This is the definition usually given in books.

Fact: If $R \to S$ is (formally) smooth, then it is flat.

Smoothness and the Module of Differentials

Lemma 1 *Consider $R \to T \to S = T/J$ and suppose that $R \to T$ is formally smooth. Then $R \to S$ is formally smooth if and only if $T/J^2 \to T/J \to 0$ splits as R-algebras.*

Note the lemma simply says that, in the case that S is a quotient T/J of a smooth R-algebra T, instead of checking *all* the diagrams in the definition of smoothness for the map $R \to S$, it suffices to check that the *one* diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d, equal] \\ T/J^2 \arrow[r] & T/J \end{tikzcd}$$

has a lift $S \to T/J^2$ for that specific S-algebra T.

Claim: If S is a finitely generated R-algebra and $R \to S$ is smooth, then $\Omega_{S/R}$ is a *projective* S-module.

Proof: Indeed, we simply apply Lemma 1 to $R \to S = T/J$, where $T = R[\underline{X}]$. By Example 1, $R \to T$ is smooth. Hence Lemma 1 says that $R \to S$ is smooth if and only if the map $T/J^2 \to T/J \to 0$ splits as an R-algebra homomorphism. As stated earlier, this is true if and only if the 2nd fundamental sequence $0 \to J/J^2 \to \Omega_{T/R} \otimes_T S \to \Omega_{S/R} \to 0$ of S-modules is split exact. In particular, since $\Omega_{T/R} \otimes_T S$ is a free S-module, we see that $R \to S$ is smooth if and only if $\Omega_{S/R}$ is a projective S-module *and* the map $\bar{d}\colon J/J^2 \hookrightarrow \Omega_{T/R} \otimes_T S$ is injective. $\square$

In fact, we will see below that when $\Omega_{S/R}$ is projective (i.e., locally free) *and* of the expected rank (i.e., there are not too many differential directions), then $R \to S$ is smooth (in that case $\bar{d}\colon J/J^2 \hookrightarrow \Omega_{T/R} \otimes_T S$ is forced to be injective).

Smoothness and Regularity

In the most standard sort of geometric situation, smoothness over a coefficient ring is equivalent to regularity.

Proposition 2 *Let k be a perfect field. Let S be a finitely generated k-algebra and Q be a maximal ideal in S. The following are equivalent.*
1. *$k \to S_Q$ is smooth.*
2. *S_Q is a regular local ring.*
3. *$(\Omega_{S/R})_Q$ is a free S_Q-module of rank* $\dim S_Q$.

However, in general the condition of smoothness is stronger than that of regularity.

Proposition 3 *Let k be any field and S a finitely generated k-algebra. The following are equivalent.*
1. *$k \to S_Q$ is smooth.*
2. *For any field $L \supseteq k$, $L \otimes_k S$ is regular (i.e., S is "geometrically regular").*

The following result gives a another case showing that smoothness over the appropriate "nice" ring is stronger than regularity. It is a corollary of the previous result when R is a field.

Corollary 4 *Let R is a regular local ring and S be a finitely generated R-algebra. If S is smooth over R, then S is regular.*

The following result provides the full interpretation of smoothness in terms of the sheaf of differentials.

Proposition 5 *Let S be a finitely generated R-algebra. Write $S \cong R[\underline{X}]/J$ where $\underline{X}$ denotes a finite set of variables $X_1, \ldots, X_n$. For a maximal ideal Q in S, define $h(Q)$ to be the minimal number of generators of the ideal $JR[\underline{X}]_{\tilde{Q}}$, where $\tilde{Q}$ is the contraction of Q to $R[\underline{X}]$. The following are equivalent.*
1. *$R \to S$ is smooth*
2. *For all maximal ideals $Q \subseteq S$, $(\Omega_{S/R})_Q$ is a free S-module of rank $n - h(Q)$.*

To illustrate these results (and especially the difference between regularity and smoothness), we now give some examples.

Example 1: [S **is regular, but even the structure map is not smooth.**]

Let $R = \mathbb{Z}_{(p)}$, $S = (\mathbb{Z}[X,Y]/(p - XY))_{(X,Y,p)}$. Then $\dim S = 2$. Let x and y denote the images of X and Y respectively in S. Since $p = xy$, $\mathfrak{m}_S = (x, y)$. Hence S is regular. (Note, in fact, that S is a ramified regular local ring since $p \in \mathfrak{m}^2$.)

On the other hand, we have

$$\Omega_{S/R} = \mathrm{coker}([\frac{\partial f}{\partial X}\ \frac{\partial f}{\partial Y}]^T) = \mathrm{coker}([-Y\ \ -X]^T)$$

Thus $\Omega_{S/R}$ is the quotient of the free S-module on dx and dy by the submodule $(-ydx - xdy)$, and hence is not projective. In particular, $R \to S$ is not smooth.

Example 2: [S is regular and contains a field k, but the inclusion is not smooth. Thus the extra assumptions on k in Prop. 2 are necessary.]

Let $k = \mathbb{F}_p(X)$, $S = \mathbb{F}_p(X^{1/p}) \cong k[T]/(T^p - X)$, and let $k \to S$ be the inclusion. Now, S is regular since it is a field, but it is not smooth over k. To see this, we will contradict condition (2) in Proposition 3. Take $L = S$. Then

$$L \otimes_k S = S \otimes_k S \cong S[U]/(U^p - X) = S[U]/(U^p - T^p) = S[U]/((U-T)^p)$$

This ring is not regular since its localisation at (T, U) is not reduced.

Example 3: [Projectivity of $\Omega_{S/R}$ is not enough for smoothness.]

Consider $R = \mathbb{Z}/2\mathbb{Z}$ and $S = \mathbb{Z}/2\mathbb{Z}[X]/(X^2)$, and let $R \to S$ be the obvious inclusion. Note that $\Omega_{S/R}$ is projective, but not of the correct rank:

$$\mathrm{rank}\Omega_{S/R} = 1 \neq 0 = \mathrm{dim} S$$

Indeed we also see that, by Proposition 2, since S is not regular, $R \to S$ is not smooth.

References

[1] Bruns, W., Herzog, J., *Cohen-Macaulay Rings,* Cambridge University Press.

[2] Matsumura, H., *Commutative ring Theory,* Cambridge University Press.

[3] Miller, C., *The Frobenius Endomorphism and Homological Dimensions,* Contemp. Math., 2003.

[4] Peskine, C., Szpiro, L., *Dimension projective finie et cohomologie locale,* IHES, 1973.