WEAKLY FUNCTORIAL STUFF

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Throughout (R,P) is an excellent local domain and R^+ is the integral closure of R in an algebraic closure K of the quotient field of R. If $x_1, \ldots, x_n \in R$ generate an ideal of height n, then $x_1, \ldots, x_n$ is called a set of parameters in R. As usual, the set is called a complete system of parameters if n is the dimension of R.

Definition. An extended valuation v on (R, P) is a rank one valuation on the quotient field of R^+/Q for some prime ideal Q of R^+ satisfying $v(x) > 0$ for all $x \in P$.

Definition. Let $(R, P) \to (S, Q)$ be a local homomorphism of complete local domains. We may extend this map to an R-algebra homomorphism θ from R^+ to S^+ by mapping the roots of a monic polynomial over R to the roots of the image polynomial over S. The choice of θ is not unique but we fix a choice once and for all. Now let v be any extended valuation on (S, Q). By restriction, v induces an extended valuation on (R, P). We will call both extended valuations v and say v is a compatible valuation on R and S.

In the quest for a mixed characteristic analog of tight closure, the precise definition for the closure seems unclear. We would like theorems which assert that if our closure has the colon-capturing property, other good results follow. To circumvent the vagueness, it seems beneficial to define a comparatively large closure operation - one that will have the colon-capturing property if other reasonable choices do. If we can show that demonstrating the colon-capturing property for this larger closure implies the desired results, we can also obtain the results for smaller closures.

Definition. Let I be an ideal in R, $x \in R$ and let v be an extended valuation on R. Then x is in the v-augmented closure of I (denoted I^v) provided that, for every $\epsilon > 0, t \in \mathbf{Z}^+$, there exists $d \in R^+$ with $v(d) < \epsilon$ such that $dx \in (I, P^t)R^+$.

Definition. We say that the v-augmented closure satisfies the colon-capturing property for R provided that if S is a finite integral extension of R, $x_1, \ldots, x_{k+1}$ is a set of parameters in S, and $u \in ((x_1, \ldots, x_k) :_S x_{k+1})$, then $u \in ((x_1, \ldots, x_k)S)^v$

The basic goal of this section is to show that the colon-capturing property implies the existence of weakly functorial Cohen-Macaulay algebras. In [Ho], Hochster demonstrated the existence of weakly functorial Cohen-Macaulay algebras for mixed characteristic domains of dimension at most three. Since the colon-capturing property is not known at this time for any of the potential closures for higher dimensions, we cannot improve upon Hochster's result at this time. However, should colon-capturing be demonstrated, our results here will allow us to get weakly functorial Cohen-Macaulay algebras more generally. The methods are heavily based on Hochster's original proof.

We must first discuss the notion of partial algebra modifications developed by Hochster and used in [Ho]. We must revamp the notation in order to get our proofs to work, but the underlying concept remains the same. Let $X_1, \ldots, X_k$ be indeterminates and let $R[X] = R[X_1, \ldots, X_k]$. By $R[X]_{\leq N}$, we mean the R-submodule of $R[X]$ spanned by all monomials

of total degree at most N. We will refer to $R[X]_{\leq N}$ as a partial algebra over R. Likewise, any finite tensor product of such objects will be called a partial algebra. So if T is a partial algebra over R, so is $T[X]_{\leq N} = T \otimes_R R[X]_{\leq N}$. Thus a partial algebra is a submodule of a polynomial ring over R defined by some perhaps complicated bound on the degrees of the monomials which appear. Of course, to any partial algebra over R, there is naturally associated a polynomial ring over R.

Definition. Let T be a partial algebra over R, A the associated polynomial ring, and $F_1, \ldots, F_n \in T$. Then $\sum_{i=1}^n F_i T$ is called a pseudo-ideal of (A, T).

It should be noted that a pseudo-ideal is just an R-submodule of A. While the definition depends upon T and the multiplicative structure of A, a pseudoideal will typically not be a subset of T and will not have a multiplicative structure.

Definition. If T is a partial algebra over R, A the associated polynomial ring, and J a pseudo-ideal of (A, T), then (A, T, J) is called an algebra triple over R.

Next we recall the definition of an algebra modification. Let A be an R-algebra. Assume $x_1, \ldots, x_{k+1}$ is a set of parameters in R with $k \geq 0$ and suppose $u \in ((x_1, \ldots, x_k)A :_A x_{k+1})$. Letting $F = u - \sum_{i=1}^k x_i X_i$, $A' = A[X_1, \ldots, X_k]/(FA[X_1, \ldots, X_k])$ is called an algebra modification of M.

Definition. Let (A, T, J) be an algebra triple over R and let $M = T/(J \cap T)$. Assume $x_1, \ldots, x_{k+1}$ is a set of parameters in R with $k \geq 0$ and suppose $u \in T$ with its image $\bar{u} \in ((x_1, \ldots, x_k)M :_M x_{k+1})$. Let $A' = A[X_1, \ldots, X_k]$, $F = u - \sum_{i=1}^k x_i X_i$, N be a fixed positive integer, $T' = T[X_1, \ldots, X_k]_{\leq N}$, and $J' = J[X_1, \ldots, X_k]_{\leq N} + FT'$. Then (A', T', J') is called an algebra triple modification of (A, T, J).

Of course, (A', T', J') is an algebra triple. We note that in this setting, $A'/J'A'$ is an algebra modification of A/JA. With our notation, we keep track of more information and this enables us to take advantage of both the algebra modification and the finiteness of $T/(J \cap T)$.

Definition. Let (A, T, J) be an algebra triple over R. Let v be an extended valuation of R. We say (A, T, J) is v-good if for every $\epsilon > 0, t \in \mathbf{Z}^+$, we can find $d \in R^+$ with $v(d) < \epsilon$ and an R-algebra homomorphism $\phi : A \to R^+[d^{-1}]$ such that $\phi(T) \subset d^{-1}R^+$ and $\phi(J) \subset d^{-1}P^t R^+$

Lemma 0.1. *If the v-augmented closure satisfies the colon-capturing property for R, (A, T, J) is v-good, and (A', T', J') is an algebra triple modification of (A, T, J), then (A', T', J') is v-good.*

Proof. We continue with the same notation with u, F, as above. Thus we have a relation $x_{k+1}u = \sum_{i=1}^k x_i u_i + w$ with each $u_i \in T$ and $w \in J \cap T$. Now choose, if necessary $x_{k+2}, \ldots, x_n \in P$ so that $x_1, \ldots, x_n$ is a complete system of parameters.

Fix $\epsilon > 0, t \in \mathbf{Z}^+$. Choose s sufficiently large so that $P^s \subseteq (x_1^{t+1}, \ldots, x_n^{t+1})R^+$. Let $\epsilon_1 = \epsilon/2(N+2)$. Since (A, T, J) is v-good, we can find $d_1 \in R^+$ with $v(d_1) < \epsilon_1$ and a suitable $\phi_1 : A \to R^+[d_1^{-1}]$ such that $\phi_1(T) \subset d_1^{-1}R^+$ and $\phi_1(J) \subset d_1^{-1}P^s R^+$. Since $\phi_1(w) \in d_1^{-1}P^s R^+$, we get $x_{k+1}\phi_1(u) = x_1\phi_1(u_1) + \cdots + x_k\phi_1(u_k) + x_1^{t+1}r_1 + \cdots + x_n^{t+1}r_n$ in $d_1^{-1}R^+$. Multiplying through by d_1, we get $x_{k+1}d_1\phi_1(u) \in (x_1, \ldots, x_k, x_{k+1}^{t+1}, \ldots, x_n^{t+1})R^+$. Hence, for some $b \in R^+$, $x_{k+1}(d_1\phi_1(u) - bx_{k+1}^t) \in (x_1, \ldots, x_k, x_{k+2}^{t+1}, \ldots, x_n^{t+1})R^+$. By the

colon-capturing property, there exists $d_2 \in R^+$ with $v(d_2) < \epsilon_1$ such that $d_2(d_1\phi_1(u) - bx_{k+1}^t) \in (x_1, \ldots, x_k, x_{k+2}^{t+1}, \ldots, x_n^{t+1}, x_{k+1}^t)R^+$. Hence there exists $b_1, \ldots, b_k \in R^+$ such that $d_2d_1\phi_1(u) - \sum_{i=1}^k x_i b_i \in (x_1^t, \ldots, x_n^t)R^+ \subseteq P^t R^+$. We set $d_3 = d_1 d_2$ and $d = d_3^{N+2}$; clearly $v(d) < \epsilon$. Now we complete the diagram

$$
\begin{array}{ccc}
R^+[d_1^{-1}] & \rightarrow & R^+[d^{-1}] \\
\phi_1 \uparrow & & \phi \uparrow \\
A & \subset & A'
\end{array}
$$

commutatively by taking $\phi(yX_1^{f_1} \cdots X_k^{f_k}) = \phi_1(y)(d_3^{-1}b_1)^{f_1} \cdots (d_3^{-1}b_k)^{f_k}$ for any $y \in A$. It is easy to check that ϕ has all the desired properties. Certainly $\phi(T') \subset d_1^{-1}d_3^{-n}R^+ \subset d^{-1}R^+$. Also $\phi(J[X_1, \ldots, X_k]_{\leq N}) \subset d_3^{-n}\phi_1(J) \subset d_3^{-n-1}P^t R^+$, while $\phi(F) \in d_3^{-1}P^t R^+$ and so $\phi(FT') \subset d^{-1}P^t R^+$; hence $\phi(J') \subset d^{-1}P^t R^+$ as desired. □

Lemma 0.2. *Let $\theta : (R, P) \rightarrow (S, Q)$ be a local map of local rings and let v be a compatible valuation on R and S. Suppose (A, T, J) is an algebra triple over R which is v-good. Then $(A \otimes S, T \otimes S, J \otimes S)$ is v-good as an algebra triple over S.*

Proof. It is clear that $(A \otimes S, T \otimes S, J \otimes S)$ is an algebra triple over S. Let $\theta : R^+ \rightarrow S^+$ be the extension of θ implicit in the definition of v. For any $\epsilon > 0, t \in \mathbf{Z}^+$, we find the appropriate map $\phi_1 : A \rightarrow R^+[d^{-1}]$. Composing with the map which θ induces on $R^+[d^{-1}]$, we get a homomorphism $\phi : A \rightarrow S^+[(\theta(d))^{-1}]$. Clearly $\phi(T) \subset (\theta(d))^{-1}S^+$ and $\phi(J) \subset (\theta(d))^{-1}P^t S^+$. Since $S^+[(\theta(d))^{-1}], (\theta(d))^{-1}S^+$, and $(\theta(d))^{-1}P^t S^+$ are S-modules and $v(\theta(d)) = v(d)$, ϕ induces an S-module homomorphism on $A \otimes S$ which has all the desired properties. □

Theorem 0.3. *Let $R \rightarrow S$ be a local homomorphism of complete local domains. Let v be a compatible valuation on R and S. Further suppose the v-augmented closure satisfies the colon-capturing property for R and S. Then there is a commutative diagram:*

$$
\begin{array}{ccc}
B & \rightarrow & C \\
\uparrow & & \uparrow \\
R & \rightarrow & S
\end{array}
$$

where B is a balanced big Cohen-Macaulay algebra over R and C is a balanced big Cohen-Macaulay algebra over S.

Proof. The basic idea of the proof is the same as that used in [HH9] and [Ho] and the basic pattern dates back to the original proof of big Cohen-Macaulay modules in the equicharacteristic case. Suppose A is an R-algebra, $x_1, \ldots, x_n$ is a system of parameters in R, and $I = (x, \ldots, x_k)A$. If $x_{k+1}u \in I$ but $u \notin I$, we have a very specific obstruction to A being Cohen-Macaulay. This obstruction can be removed by forming an algebra modification of A. Take $A' = A[X_1, \ldots, X_k]/(u - \sum_{i=1}^k x_i X_i)$. Intuitively, one may simply construct a long chain of algebra modifications starting from R to obtain an R-algebra in which all of the obstructions are gone and so every system of parameters forms a regular sequence. The limit B will be a balanced big Cohen-Macaulay algebra over R unless $PB = B$ where P is the maximal ideal of R. Thus, proving the existence of B comes down to showing $1 \notin PB$. Now if the identity is in PB, the offending equation involves only finitely many elements from B and so occurs as the result of one specific modification and so the limit

process does not really play a role. More formally, in [HH9], B is constructed as the direct limit of finitely generated algebras constructed from finite sequences of modifications and it is seen that if $1 \in PB$, we actually have $1 \in PA$ where A is formed from R via a finite sequence of algebra modifications. Likewise, C is constructed as a direct limit using algebra modifications of $B \otimes_R S$. Again following [HH9], the theorem is valid unless there exists a sequence of modifications $R = T_0, T_1, \ldots, T_r, U_0 = T_r \otimes_R S, U_1, \ldots, U_s$ with $1 \in QU_s$ for Q the maximal ideal of S where each T_{i+1} (resp. U_{i+1}) is an algebra modification of T_i (resp. U_i). So we simply must show such a sequence is impossible.

Assume we have such a bad double sequence of algebra modifications. Ultimately, U_s is constructed as a homomorphic image of a polynomial ring over S. The condition $1 \in QU_r$ corresponds to an equation in the polynomial ring: $1 = \sum_{i=1}^n x_i F_i + \sum_{i=1}^m G_i H_i$ where each H_i maps to the zero element in U_s because it played the role of F in a specific algebra modification. Now each modification was performed because of a relation which can be lifted to a relation in the polynomial ring of the form $y_{k+1} u = \sum_{i+1}^k y_i u_i + \sum_{i=1}^j G_{li} H_i$ where the y's and u's vary from modification to modification. There is clearly some bound for the degree of the polynomials $F_i, G_i H_i, G_{li} H_i, u, u_i$. and so polynomials of sufficiently large degree add nothing to the process. Accordingly, Hochster introduced partial algebra modifications in [Ho] and noted that it was sufficient to prove that there are no bad partial algebra modifications.

Thus far, this is just Hochster's proof worded differently. At this point the proofs diverge. Let $R = T_0, T_1, \ldots, T_r, U_0 = T_r \otimes_R S, U_1, \ldots, U_s$ be a bad sequence of algebra modifications. Then we have a corresponding bad sequence of algebra triple modifications $(R, R, (0)), (A_{11}, T_{11}, J_{11}), \ldots, (A_{1r}, T_{1r}, J_{1r}), (A_{1r} \otimes S, T_{1r} \otimes J_{1r}, J \otimes S), (A_{21}, T_{21}, J_{21}), \ldots, (A_{2s}, T_{2s}, J_{2s})$. The equation $1 = \sum_{i=1}^n x_i F_i + \sum_{i=1}^k G_i H_i$ immediately gives, as a relation in T_{2s}, that $1 \in QT_{2s} + J_{2s}$ since each $G_i H_i$ is in J_{2s}. Next the algebra triple (R,R,(0)) is trivially v-good and repeated application of the lemmas implies (A_{2s}, T_{2s}, J_{2s}) is v-good. Choose $\epsilon = v(Q)$ and $t = 1$. We then find $d \in S^+$ with $v(d) < \epsilon$ and a homomorphism $\phi A_{2s} \to S^+[d^{-1}]$ such that $\phi(T_{2s}) \subset d^{-1} S^+$ and $\phi(J_{2s}) \subset d^{-1} Q S^+$. Applying ϕ to our bad relation gives $1 \in Q d^{-1} S^+ + d^{-1} Q S^+$. Hence $d \in Q S^+$. But $v(d) < v(Q)$, a contradiction which proves the theorem. $\qquad \square$