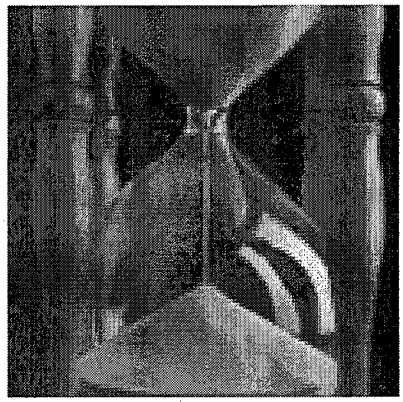
The Geometry of Piles of Salt

Thinking Deeply About Simple Things



University of Utah
Teacher's Math Circle

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By
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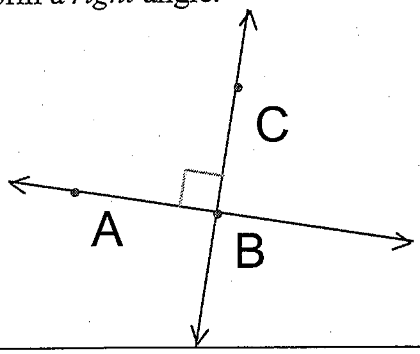
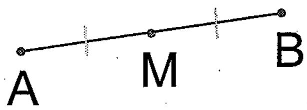
Important Terms (the word line may be replaced by the word segment or ray in any of the following definitions)

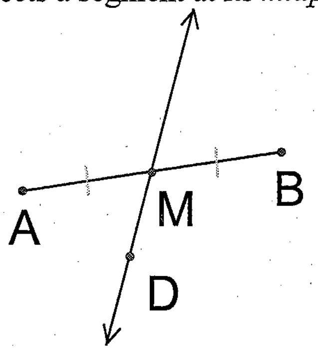
Perpendicular: Two lines are perpendicular if they intersect and form a *right* angle.



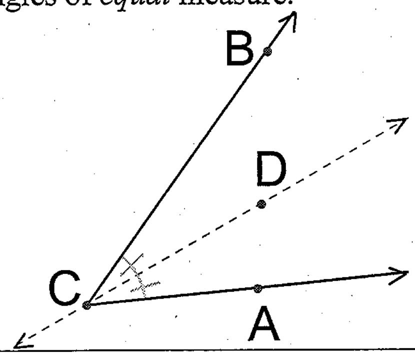
Midpoint of a segment: A point on a segment that is *equidistant* from the endpoints of the segment.



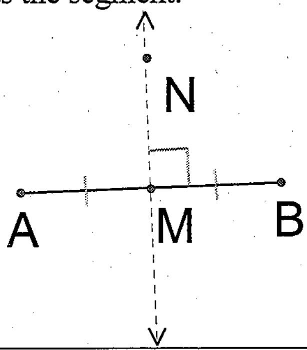
Bisector of a segment: A line that intersects a segment at its *midpoint*.



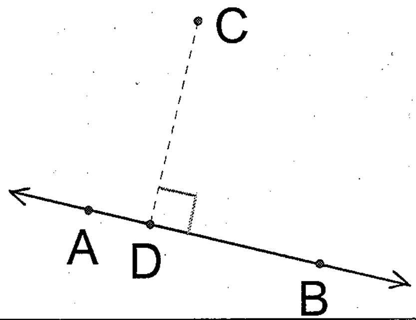
Bisector of an angle: A line through the vertex of an angle that divides the angle into two smaller angles of *equal* measure.



Perpendicular bisector of a segment: A line that is perpendicular to a segment *and* bisects the segment.

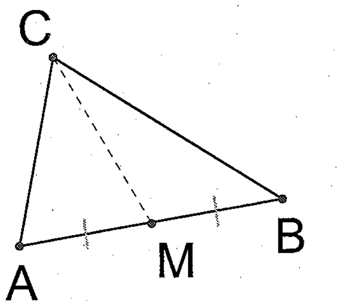


Distance from a point to a line: The distance measured along a *perpendicular* segment from the point to the line.

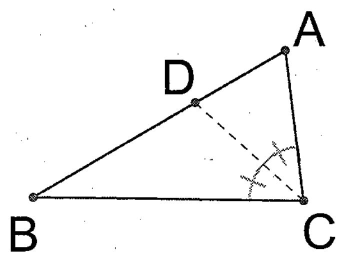


Special segments in a triangle:

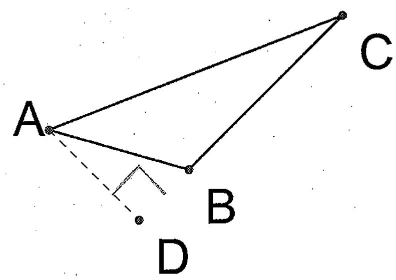
Median: A segment connecting a *vertex* of a triangle with the *midpoint* of the opposite side.



Angle bisector: A segment from a *vertex* of a triangle to a point on the opposite side, which *bisects* the angle.



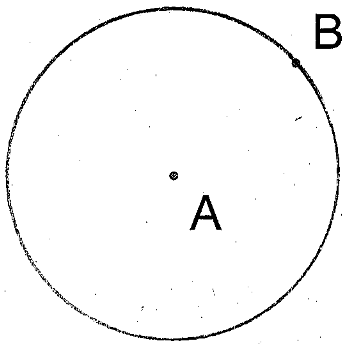
Altitude: A segment from a *vertex* of a triangle which is *perpendicular* to the opposite side. The foot of the altitude is the intersection point of the altitude and the side.



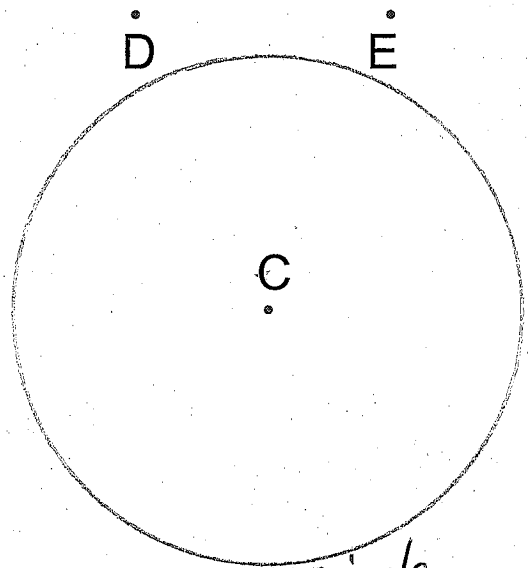
There is a difference between *sketching*, *drawing*, and *constructing* an object. When you *sketch* an equilateral triangle, you make a freehand sketch that looks equilateral. When you *draw* an equilateral triangle, you may use tools like a ruler, straightedge, t-square, protractor, and templates to accurately measure and render an equilateral triangle with straight, equal length sides and 60° angles. When you *construct* an equilateral triangle, you may not measure with a ruler or a protractor. It is a precise way of drawing using only a compass and straightedge and following specific rules. The first rule is that a point must either be given or be the intersection of figures that have already been constructed. The second rule is that a straightedge can draw the line through two points, and the third rule is that a compass can draw a circle with center at one point and passing through a second point, or with a given radius.

We define a *locus* as a set of points that satisfy a certain set of conditions.

1. Construct the locus of points that are a distance of AB units away from point A.



2. Construct the locus of points that are a distance of DE units away from point C.



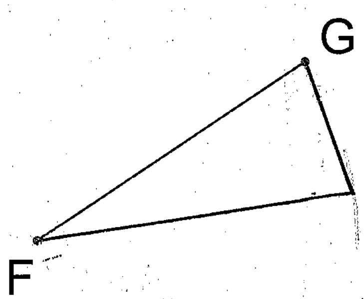
3. What is the common name for these *loci* (plural of locus) that we constructed? circle

4. Point A (and C) in the locus construction above is called the center of the circle.

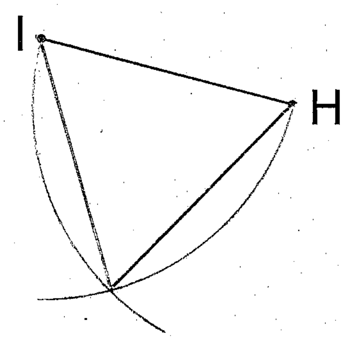
5. The distance AB (and DE) in the locus construction above is called the radius of the circle.

6. The *locus* definition of a circle is: A *circle* is the locus of all points a given distance (the radius) away from a given point (the center).

7. Construct an *isosceles* triangle using segment FG as a leg.



8. Construct an *equilateral* triangle using segment IH as a side.

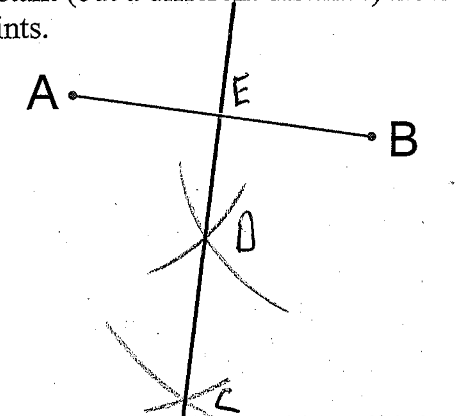


An alternative to the definition of the perpendicular bisector of a segment that was given on page 1 is the locus definition for the perpendicular bisector of a segment. This locus definition relies on a property of perpendicular bisectors stated in the *Perpendicular Bisector Theorem*. The *Perpendicular Bisector Theorem* says that any point on the perpendicular bisector of a segment is equidistant from the *endpoints* of the segment. Since the converse (any point that is equidistant from the endpoints of a segment is on the perpendicular

~~bisector of the segment) is~~ also true, we may define the perpendicular bisector of a segment as the locus of all points equidistant from the endpoints of a segment.

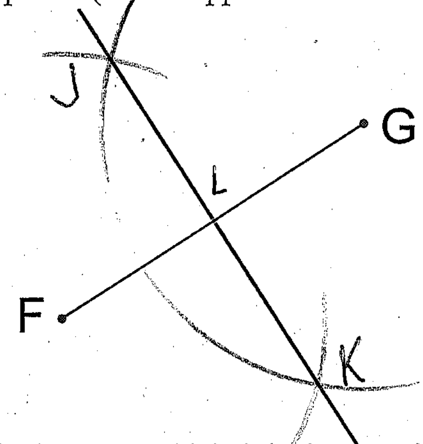
One of the most important constructions is that of the perpendicular bisector ($\perp$ bisector) of a segment.

9. Construct a point *C*, *below* segment *AB*, that is equidistant from the endpoints. Then construct a second point *D*, *below* segment *AB*, that is also equidistant (but a different distance) from the endpoints.



Draw the line *CD* and extend it until it intersects segment *AB*. Label this intersection point *E*.

10. Construct two points *J* and *K*, one above and one below segment *FG*, that are equidistant from the endpoints, and both the same distance away from the endpoints (but on opposite sides of segment *FG*).



Draw the line *JK* and label the intersection point *L*.

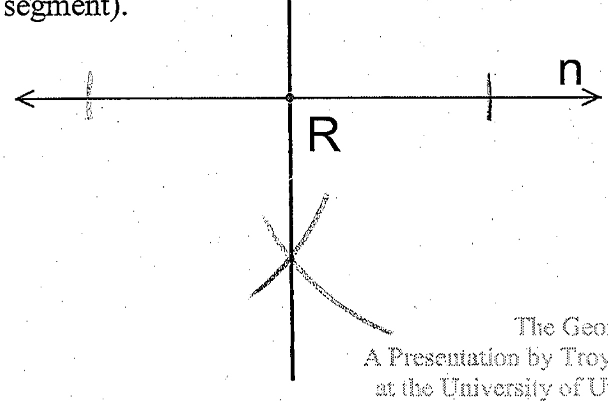
We only needed to construct two points equidistant from the endpoints of the segment in order to construct the $\perp$ bisector of the segment, although we could construct several more to verify that they are all collinear.

A bonus that comes from constructing the $\perp$ bisector of a segment is that we construct the midpoint of the segment. In the constructions above, *E* is the midpoint of segment *AB*, and *L* is the midpoint of segment *JK*.

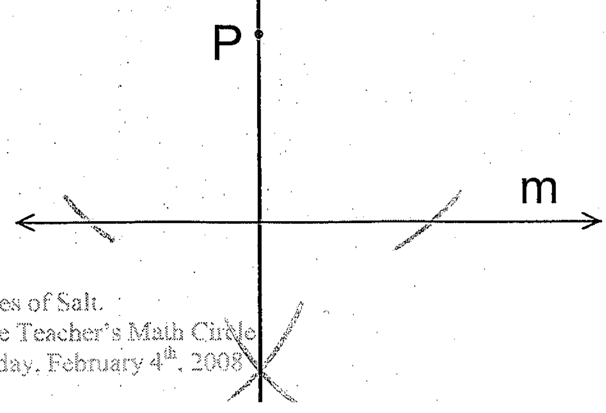
The most efficient *algorithm* for constructing the $\perp$ bisector of a segment (and also its midpoint) is to open your compass greater than half the length of the segment and swing an arc from one endpoint that extends both above and below the segment, then with the same radius swing an arc from the other endpoint that intersects the first arc in two places, both above and below the segment, then draw the line through the two intersection points. This line will be the locus of all points equidistant from the endpoints of the segment, and hence its perpendicular bisector. Of course, if the segment is close to the edge of the paper and there is not enough room to construct points on both sides of the segment, you could just construct two different points on the same side.

The construction of the $\perp$ bisector is a building block for many other constructions.

11. Construct the line through point *R* that is *perpendicular* to line *n* (hint: construct a segment on line *n* such that *R* is the midpoint of the segment).



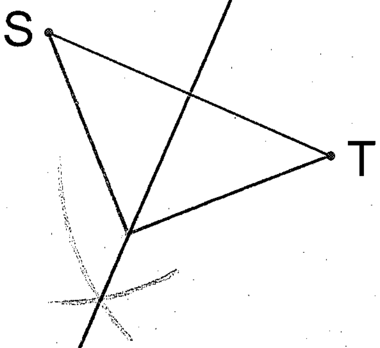
12. Construct the line through point *P* that is *perpendicular* to line *m* (hint: construct a segment on line *m* such that *P* is equidistant from the endpoints of the segment).



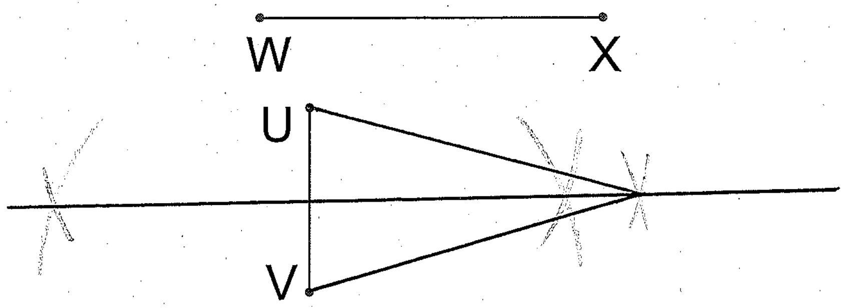
13. ~~Provide a convincing~~ argument to another person in your group about why the construction in problem number **11** works.

14. Provide a convincing argument to a *different* person in your group about why the construction in problem number **12** works.

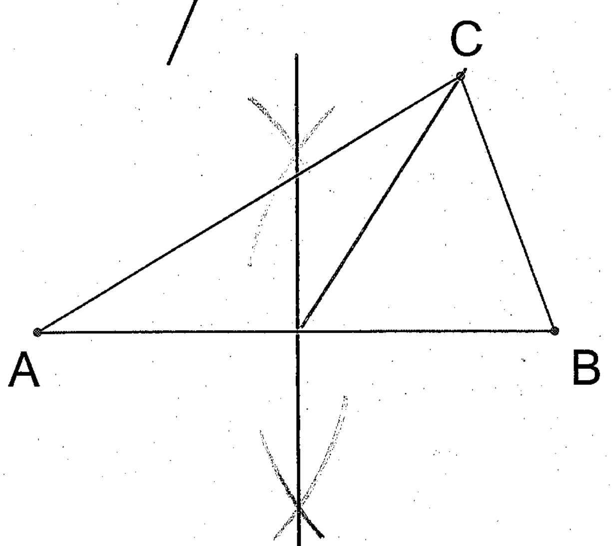
15. Construct an *isosceles* triangle using *ST* as a *base.*



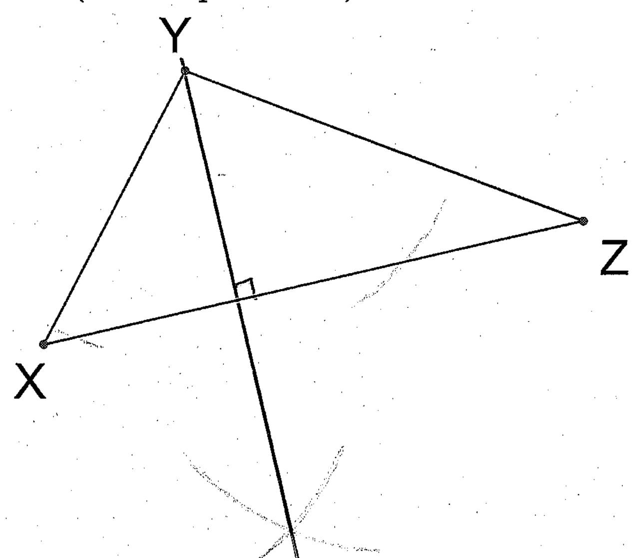
16. Construct an *isosceles* triangle using *UV* as a *base* and *WX* as a *leg.*



17. Construct the *perpendicular bisector* of side *AB* of triangle *ABC*, and the *median* from vertex *C.*

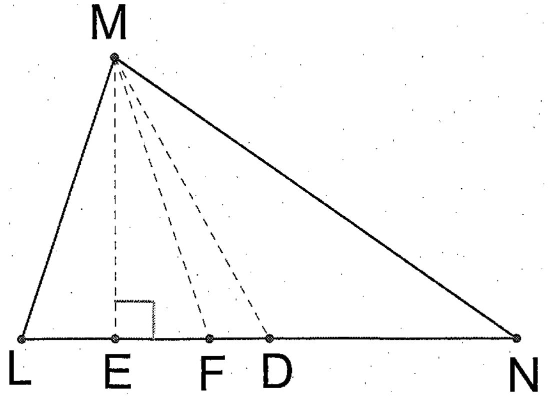


18. Construct the *altitude* from vertex *Y* in triangle *XYZ* (hint: see problem **12**).

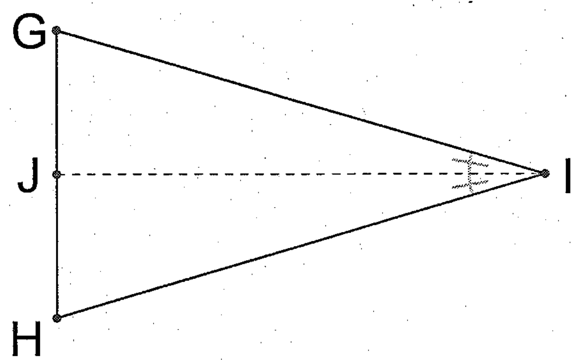


Before we learn the algorithm for bisecting an angle, let's explore the properties of *medians*, *altitudes*, and *angle bisectors* in *scalene* and *isosceles* triangles.

In *scalene* triangle *LMN*, *ME* is an *altitude*, *MF* is an *angle bisector*, and *MD* is a *median.*



In *isosceles* triangle *GHI*, *IJ* is an *angle bisector* of *vertex* angle *I.*



~~Notice in a scalene triangle~~ that the *altitude*, *angle bisector*, and *median* constructed from the same vertex are all different segments. The "feet" of these segments will always be in the same order, with the *angle bisector* between the *altitude* and the *median*.

19. Using isosceles triangle *GHI* with angle bisector *IJ* from the bottom of page 4, prove that triangle *GJI* is congruent to triangle *HJI*.

Since $\overline{IJ}$ bisects $\angle I$, and $GI = IH$ isosceles and $IJ = IJ$, $\triangle GJI \cong \triangle HJI$ by SAS

20. After proving the triangles are congruent, what can you say about point *J*? midpoint Why?
$GJ = HJ$, corresponding parts of $\cong$ triangles

21. What can you say about the measures of angles *GJI* and *HJI*? right angles Why?
both congruent and supplementary

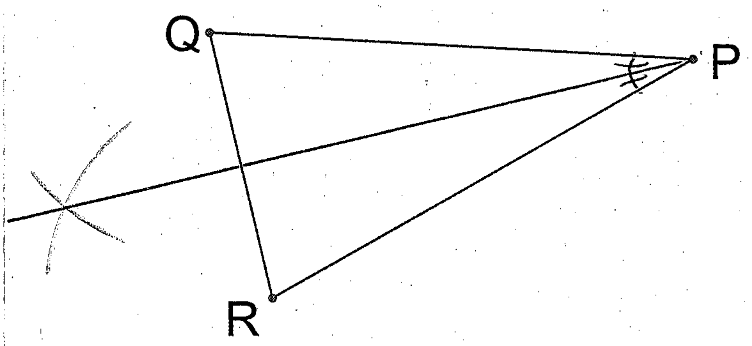
22. Using information from problem number **20**, what is another name for segment *IJ*? median

23. Using information from problem number **21**, what is a different name for segment *IJ*? Altitude
(with respect to vertex *I*)

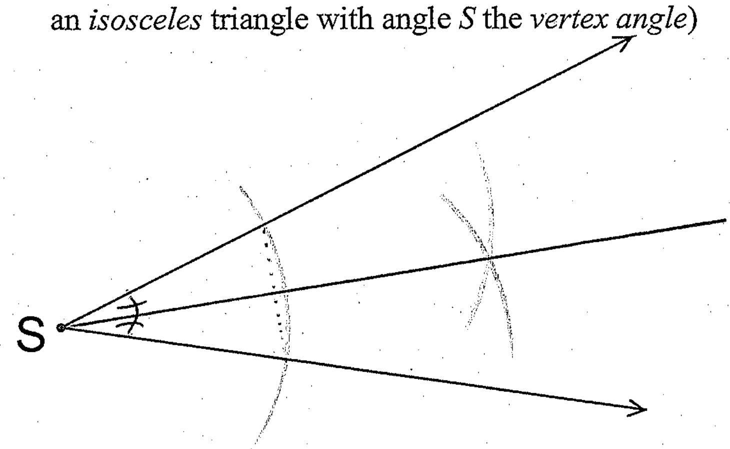
24. Combining the properties of the segments in problems **22** and **23**, what is another name for the line containing segment *IJ* (with respect to side *GH*)? $\perp$ bisector

The preceding properties are summarized in the Isosceles Triangle Vertex Angle Bisector Theorem: In an *isosceles* triangle, the *bisector* of the *vertex angle* is also the *altitude* and *median* to the base (and the perpendicular bisector of the base). In other words, all four segments determine the same line. We can use this fact specifically to construct the angle bisector of the vertex angle in an *isosceles* triangle, and more generally, to bisect any angle.

25. Construct the *angle bisector* of *vertex angle P* in *isosceles* triangle *PQR* by constructing the $\perp$ bisector of *base QR* (you already know one point on the $\perp$ bisector of *QR*).



26. Construct the *bisector* of angle *S* (first, construct an *isosceles* triangle with angle *S* the *vertex angle*)



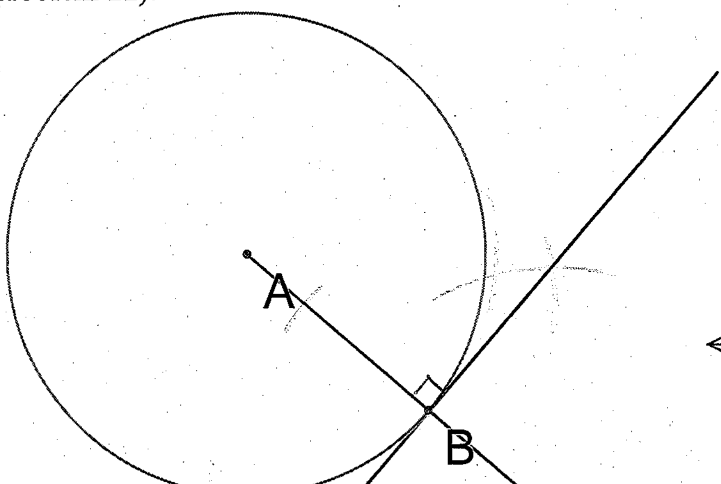
~~We now have an~~ *~~algorithm~~* ~~for~~ constructing the angle bisector. With your compass at the vertex of the angle, swing an arc that crosses both sides of the angle. The intersection points will be the vertices of the base angles of an isosceles triangle (you don't actually need to draw the base). Construct the $\perp$ bisector of this base by constructing a second point equidistant to the endpoints of the base. The line connecting this point with the vertex will bisect the angle.

An important theorem that we will use when talking about piles of salt is the *Angle Bisector Theorem*, which says that any point on the bisector of an angle is equidistant from the *sides* of the angle. The converse of this theorem says that any point which is equidistant from the sides of an angle is on the angle bisector.

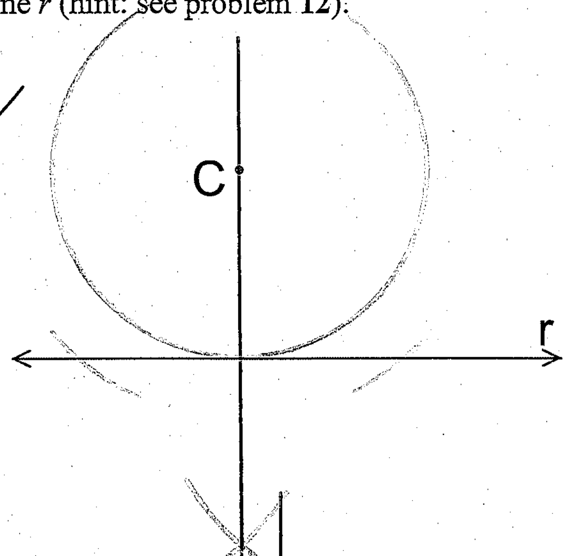
27. Using the Angle Bisector Theorem, write a locus definition for the angle bisector. The bisector of an angle is the locus of points equidistant from the sides of an angle

Important theorem: A *tangent* line to a circle is perpendicular to the radius of the circle at the point of tangency.

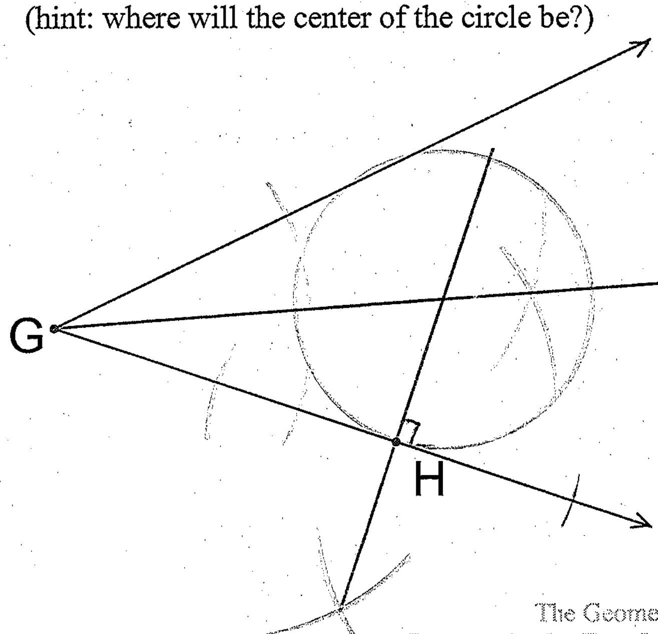
28. Construct the *tangent* to circle A at point B (hint: see problem **11**).



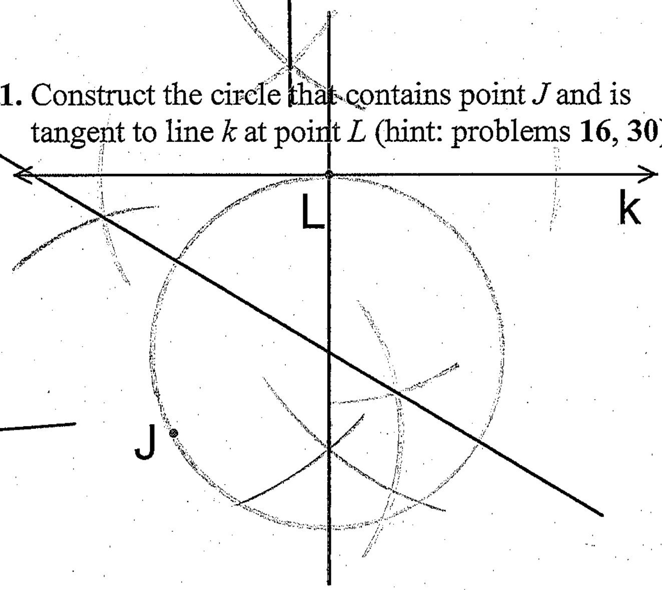
29. Construct the circle with center C and tangent to line r (hint: see problem **12**).



30. Construct the circle *inscribed* in angle G (tangent to both sides) that is tangent to side GH at point H. (hint: where will the center of the circle be?)



31. Construct the circle that contains point J and is tangent to line k at point L (hint: problems **16**, **30**).



Salt Pile Experiment

The *angle of repose* is the minimum angle at which a granular material can no longer support itself, and will flow under the influence of gravity. (The term "granular" covers a wide range, since even large boulders that accumulate at the foot of a mountain have an angle of repose, and a rockslide or avalanche occurs if this angle is exceeded.) The angle of repose is the measure of the angle between the surface of the material and horizontal.

The steepness of the angle of repose is affected by such properties as the size and angularity of the grains, density of the grains, cohesion between the grains (due to electrostatic energy, magnetism, water film, etc.), substrate roughness, shear-stress, and gravity. Ordinary table salt has an angle of repose of about 32°.

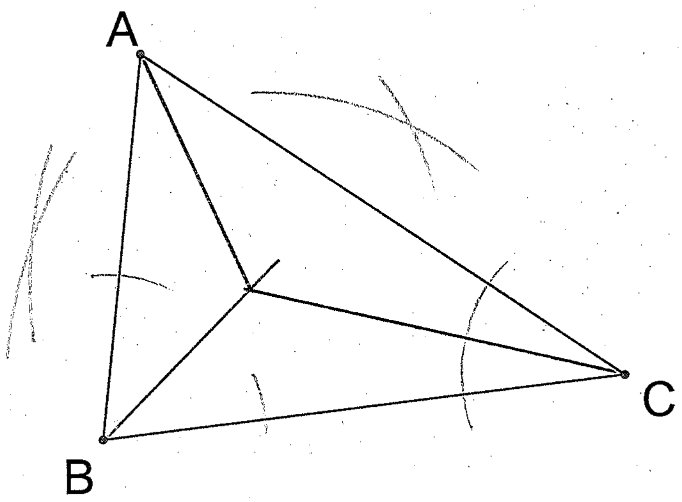
Circle Experiment

- Predict what will happen when salt is poured on a circular shape.
  It will make a cone.
- Pour salt on circular shape and observe if your predictions are verified.
  yes.
- Where is the apex of the cone in relation to the circular base?
  directly above the center of the circle that is the base.
- If you know the diameter of the circular base, and the angle of repose, can you calculate the height?
  yes h θ r $r = \frac{d}{2}$ $\tan\theta = \frac{h}{r} \rightarrow h = r\tan\theta$

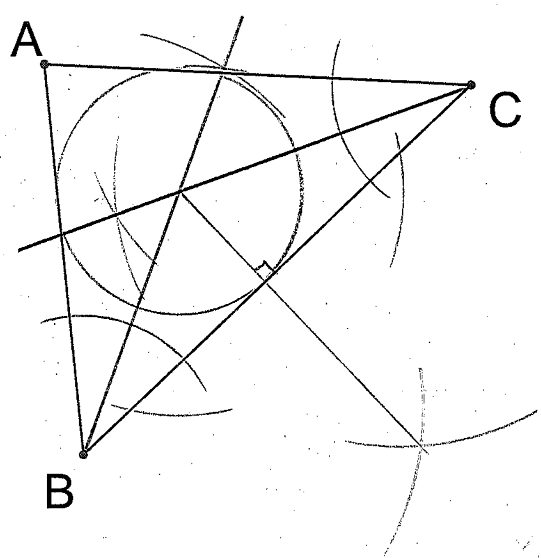
Triangle Experiment

- Predict what will happen when salt is poured on a triangular shape.
  It will form a triangular pyramid.
- Pour salt on triangular shape and observe if your predictions are correct.
  yes
- Where is the apex of the pyramid in relation to the triangular base?
  directly above the .....
- What are the ridgelines in relation to the triangular base?
  Angle bisectors (equidistant from sides), concur at incenter

Construct a model of your observations from the triangular experiment.



Inscribe a circle in the triangle (Don't forget to construct the point of tangency of the incircle).



Quadrilateral Experiment

- Predict what will happen when salt is poured on a quadrilateral shape.

  It will form a quadrilateral pyramid?

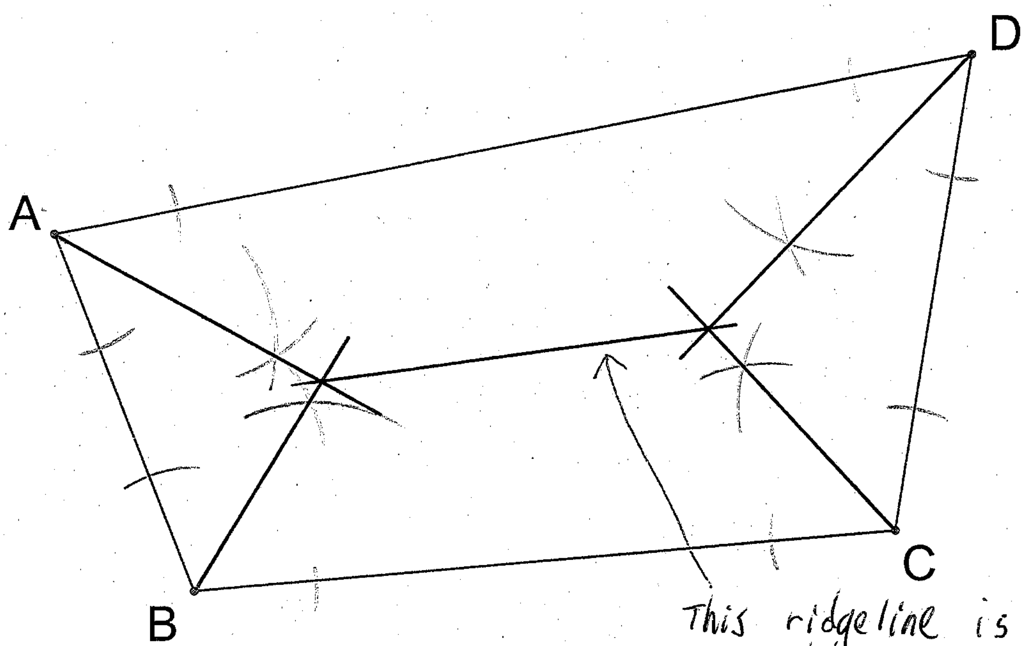
- Pour salt on quadrilateral shape and observe if your predictions are correct.

  I was wrong.

- Describe the ridgelines in relation to the quadrilateral base?

  They are the bisectors of the angles determined by the sides

Construct a model of your observations from the triangular experiment.



This ridgeline is equidistant from sides $\overline{AD}$ and $\overline{BC}$. (It bisects the angle formed by the extensions of these sides)

Circle Near Edge Experiment

- Predict what will happen when salt is poured on a surface with a hole near the edge.

  It will form a part of a cone with a hole in it.

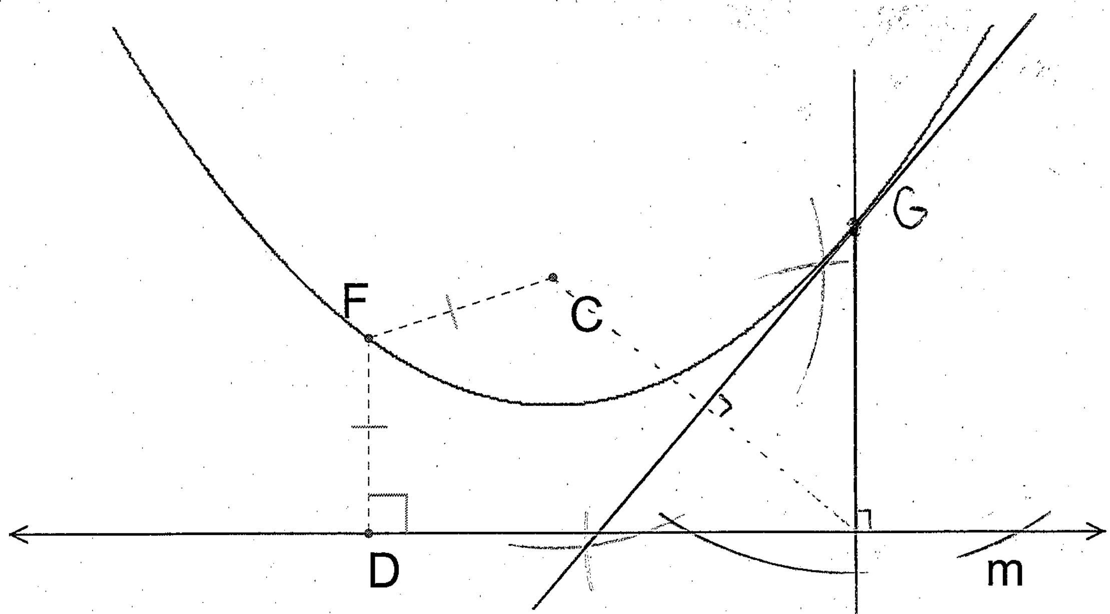
- Pour salt on surface and observe if your predictions are correct.

  Cool!

- Describe the ridgeline in relation to the edge and the circle.

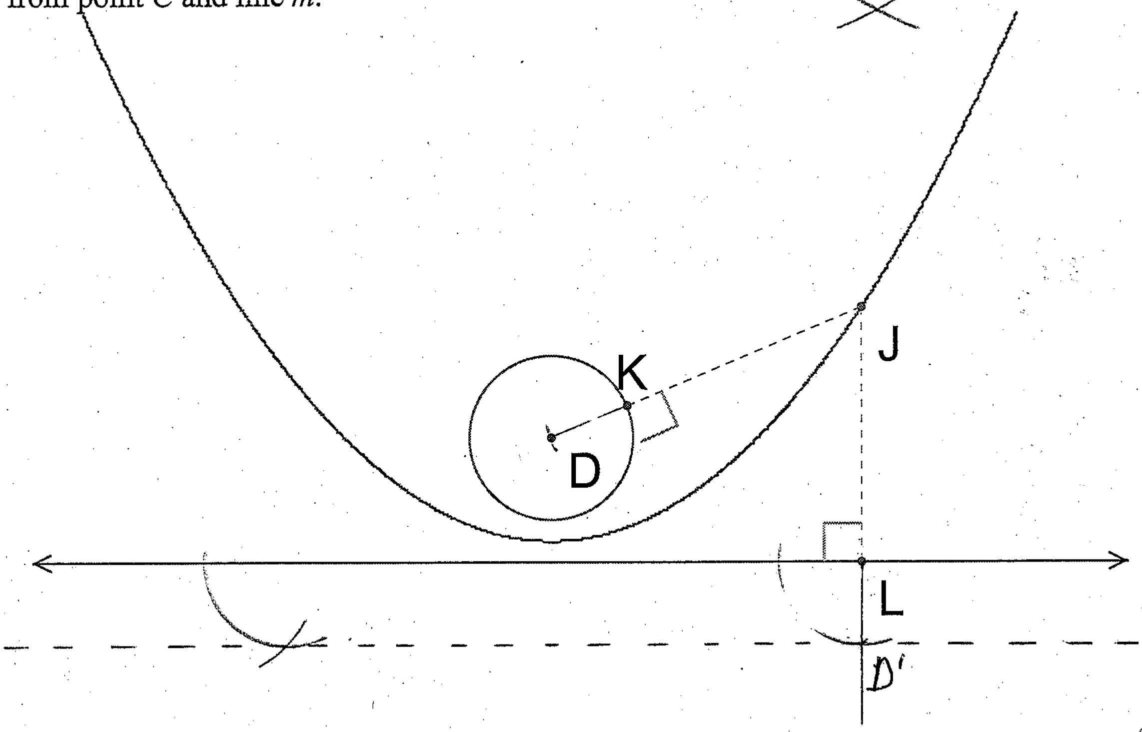
  It appears to be a parabola

~~Definition: A *parabola* is the~~ locus of points equidistant from a *point* (focus) and a *line* (directrix).



The distance from F to line m is measured along the perpendicular from F to line m. This distance FD is the same as the distance FC. Since F is equidistant from points C and D, describe how to construct point F given point D on line m and point C. Construct the ⊥ bisector of segment $\overline{CD}$, and construct the ⊥ to line m through D, F will be the intersection of these lines

Choose another point G on the parabola, and perform the construction that would demonstrate that G is equidistant from point C and line m.



Why is the distance from J to K the same as the distance from J to L? From the salt experiment, The salt slides off the edges so that the ridgeline that is formed is equidistant from the edges

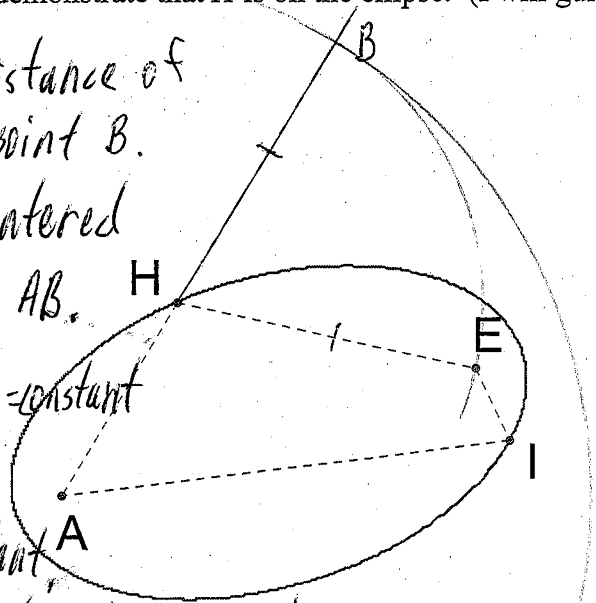
The definition of a *parabola* is the locus of points that are equidistant from a *point* and a line, not a *circle* and a line. Can you prove that the locus is a parabola? in this experiment Extend line JL below line L a distance of DK. Name the point D'. Therefor, DK + KL = D'L + LJ, or DJ = D'J. Construct a line through D' parallel to Line L. This can be done for any point on the ridgeline so the ridgeline is the locus of points equidistant from point D and line D', so it is a parabola.

Definition: An *ellipse* is the locus of points whose *sum* to *two* fixed *points* (foci) is *constant*.

$HA + HE = IA + IE = constant$

Perform the construction that will demonstrate that H is on the ellipse. (I will guide you)

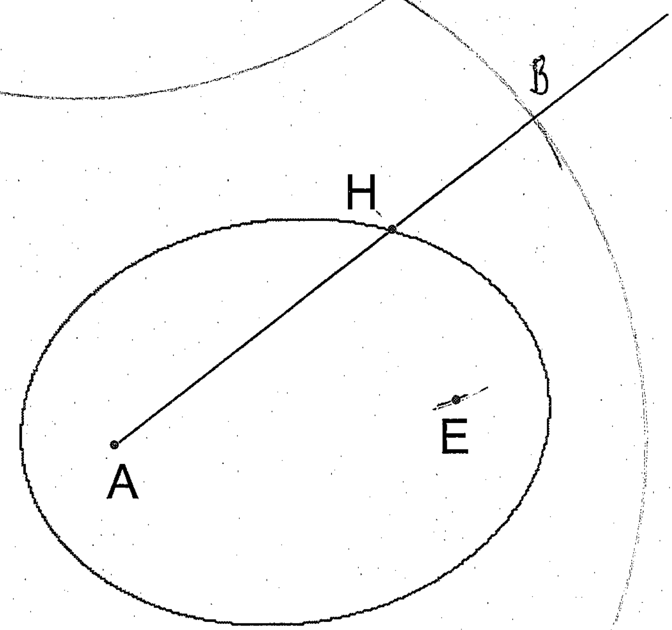
Extend line AH a distance of HE and name the endpoint B. Construct the circle centered at A, with radius AB.



AH + HB = AH + HE = AB = constant

as H moves around the ellipse, AB remains constant, as does AH+HE, so the locus of H is an ellipse

Practice the construction again.

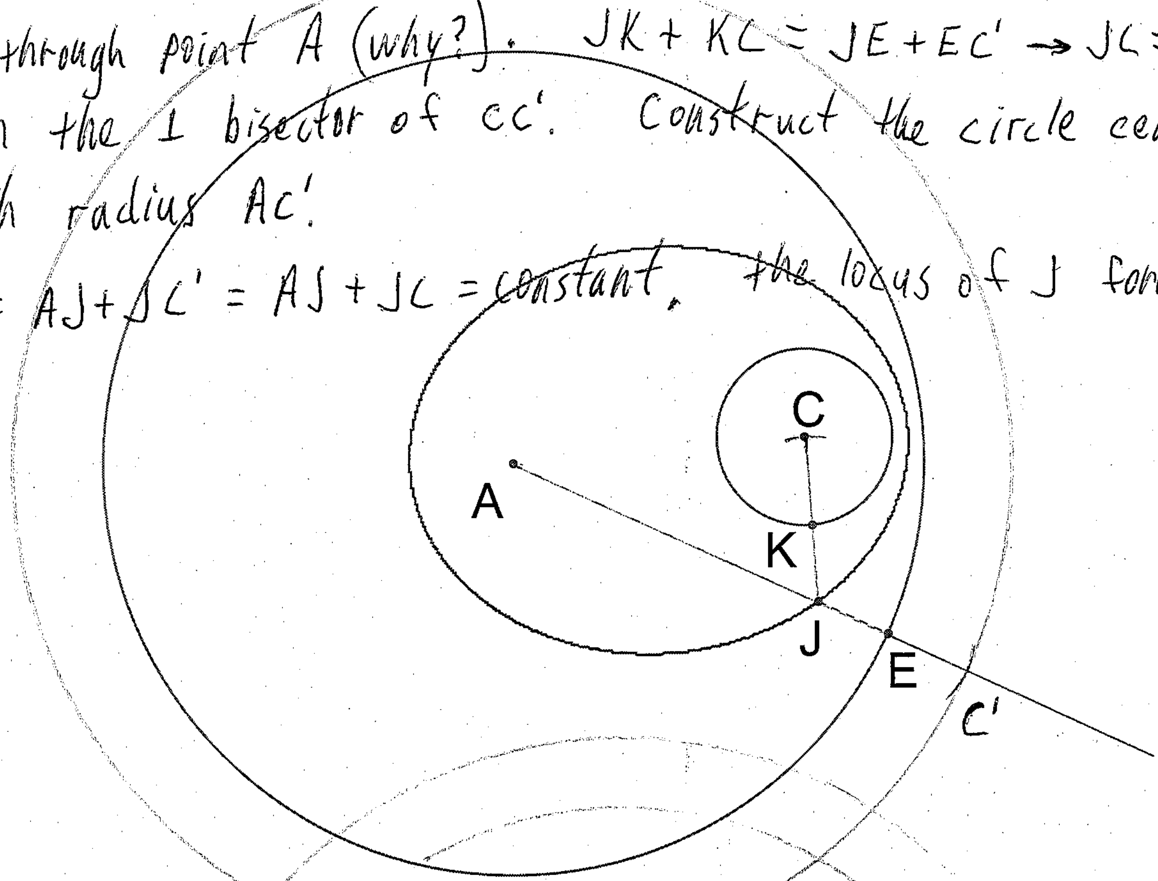


Why is the distance from J to K the same as the distance from J to E? From the salt experiment. The salt slides off the edges, forming a ridgeline equidistant from the edges

The definition of an *ellipse* is the locus of points whose *sum* to *two* fixed *points* is *constant*. Can you prove that the locus of J is an *ellipse*?

Extend line JE a distance of KC and name the endpoint C'. Line JE also goes through point A (why?). $JK + KC = JE + EC' \rightarrow JC = JC'$, so J is on the $\perp$ bisector of CC'. Construct the circle centered at A with radius AC'.

Since $AC' = AJ + JC' = AJ + JC =$ constant, the locus of J forms an ellipse



Extend AC. Name the intersection of AC and the locus D, and the intersection of AC and circle C, B. Extend AD a distance of BC and name the endpoint F. Construct the circle centered at F with radius BC and name the intersection of this circle with line AC, E.

$FD - FE = CD - CB$

$ED = BD$

So circle centered at A with radius AE is the circle in the salt experiment, and $AD + DC = AD + DF = AF =$ constant, so the locus of D is an ellipse.