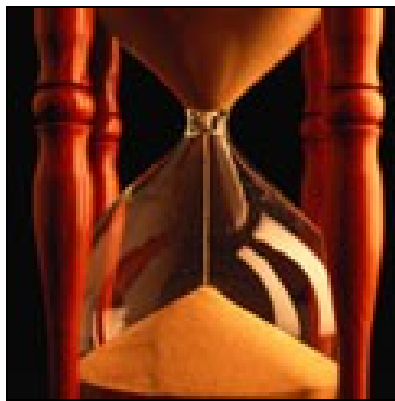


The Geometry of Piles of Salt

Thinking Deeply About Simple Things



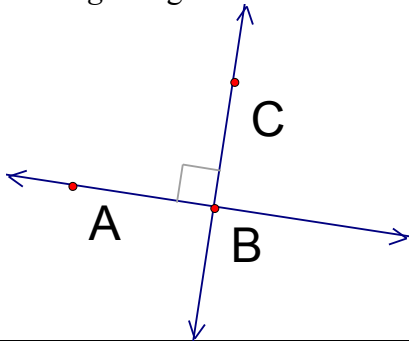
**University of Utah
Teacher's Math Circle**

Monday, February 4th, 2008

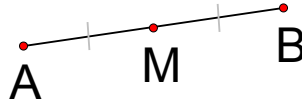
**By
Troy Jones
Waterford School**

Important Terms (the word line may be replaced by the word segment or ray in any of the following definitions)

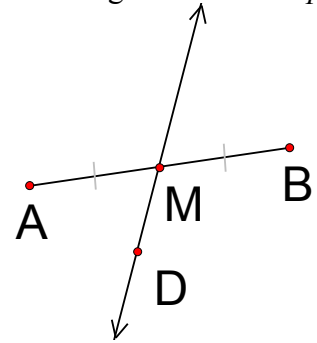
Perpendicular: Two lines are perpendicular if they intersect and form a *right angle*.



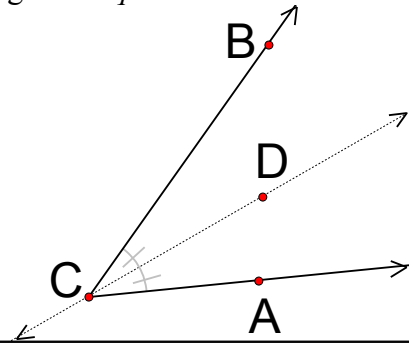
Midpoint of a segment: A point on a segment that is *equidistant* from the endpoints of the segment.



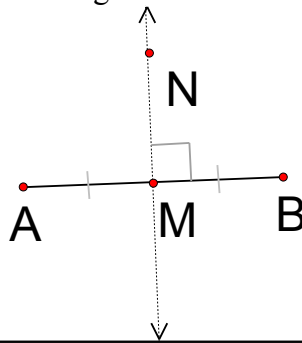
Bisector of a segment: A line that intersects a segment at its *midpoint*.



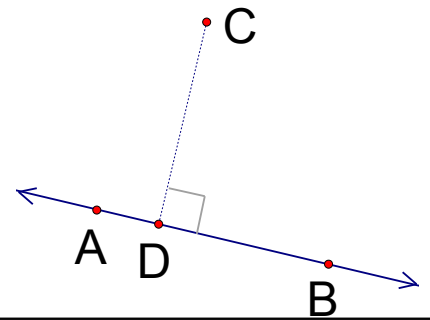
Bisector of an angle: A line through the vertex of an angle that divides the angle into two smaller angles of *equal* measure.



Perpendicular bisector of a segment: A line that is perpendicular to a segment and bisects the segment.

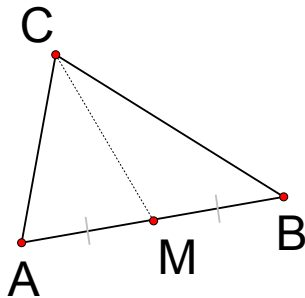


Distance from a point to a line: The distance measured along a *perpendicular* segment from the point to the line.

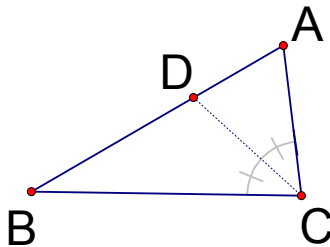


Special segments in a triangle:

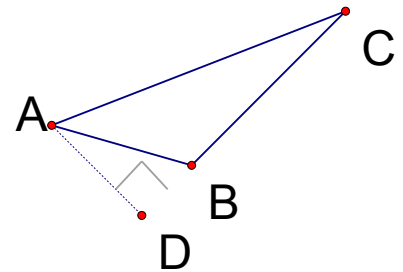
Median: A segment connecting a *vertex* of a triangle with the *midpoint* of the opposite side.



Angle bisector: A segment from a *vertex* of a triangle to a point on the opposite side, which *bisects* the angle.



Altitude: A segment from a *vertex* of a triangle which is *perpendicular* to the opposite side. The foot of the altitude is the intersection point of the altitude and the side.



There is a difference between *sketching*, *drawing*, and *constructing* an object. When you *sketch* an equilateral triangle, you make a freehand sketch that looks equilateral. When you *draw* an equilateral triangle, you may use tools like a ruler, straightedge, t-square, protractor, and templates to accurately measure and render an equilateral triangle with straight, equal length sides and 60° angles. When you *construct* an equilateral triangle, you may not measure with a ruler or a protractor. It is a precise way of drawing using only a compass and straightedge and following specific rules. The first rule is that a point must either be given or be the intersection of figures that have already been constructed. The second rule is that a straightedge can draw the line through two points, and the third rule is that a compass can draw a circle with center at one point and passing through a second point, or with a given radius.

We define a *locus* as a set of points that satisfy a certain set of conditions.

1. Construct the locus of points that are a distance of AB units away from point A .
2. Construct the locus of points that are a distance of DE units away from point C .



3. What is the common name for these *loci* (plural of locus) that we constructed? _____
4. Point A (and C) in the locus construction above is called the _____ of the circle.
5. The distance AB (and DE) in the locus construction above is called the _____ of the circle.
6. The *locus* definition of a circle is: A *circle* is the locus of all points a given _____ (the radius) away from a given _____ (the center).
7. Construct an *isosceles* triangle using segment FG as a leg.
8. Construct an *equilateral* triangle using segment IH as a side.



An alternative to the definition of the perpendicular bisector of a segment that was given on page 1 is the locus definition for the perpendicular bisector of a segment. This locus definition relies on a property of perpendicular bisectors stated in the *Perpendicular Bisector Theorem*. The *Perpendicular Bisector Theorem* says that any point on the perpendicular bisector of a segment is equidistant from the *endpoints* of the segment. Since the converse (any point that is equidistant from the endpoints of a segment is on the perpendicular bisector of the segment) is also true, we may define the perpendicular bisector of a segment as the locus of all points equidistant from the endpoints of a segment.

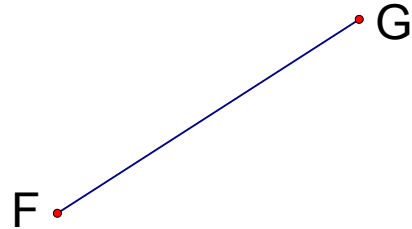
One of the most important constructions is that of the perpendicular bisector (\perp bisector) of a segment.

9. Construct a point C , below segment AB , that is equidistant from the endpoints. Then construct a second point D , below segment AB , that is also equidistant (but a different distance) from the endpoints.



Draw the line CD and extend it until it intersects segment AB . Label this intersection point E .

10. Construct two points J and K , one above and one below segment FG , that are equidistant from the endpoints, and both the same distance away from the endpoints (but on opposite sides of segment FG).



Draw the line JK and label the intersection point L .

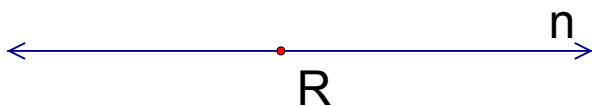
We only needed to construct two points equidistant from the endpoints of the segment in order to construct the \perp bisector of the segment, although we could construct several more to verify that they are all collinear.

A bonus that comes from constructing the \perp bisector of a segment is that we construct the midpoint of the segment. In the constructions above, E is the midpoint of segment AB , and L is the midpoint of segment JK .

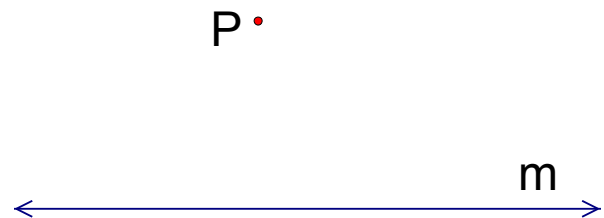
The most efficient *algorithm* for constructing the \perp bisector of a segment (and also its midpoint) is to open your compass greater than half the length of the segment and swing an arc from one endpoint that extends both above and below the segment, then with the same radius swing an arc from the other endpoint that intersects the first arc in two places, both above and below the segment, then draw the line through the two intersection points. This line will be the locus of all points equidistant from the endpoints of the segment, and hence its perpendicular bisector. Of course, if the segment is close to the edge of the paper and there is not enough room to construct points on both sides of the segment, you could just construct two different points on the same side.

The construction of the \perp bisector is a building block for many other constructions.

11. Construct the line through point R that is *perpendicular* to line n (hint: construct a segment on line n such that R is the midpoint of the segment).



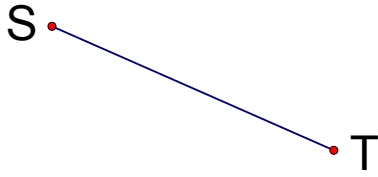
12. Construct the line through point P that is *perpendicular* to line m (hint: construct a segment on line m such that P is equidistant from the endpoints of the segment).



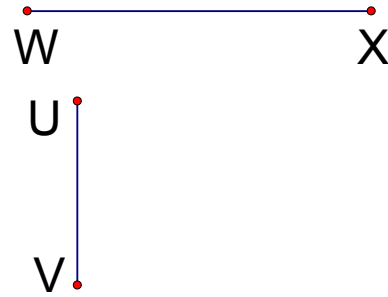
13. Provide a convincing argument to another person in your group about why the construction in problem number 11 works.

14. Provide a convincing argument to a *different* person in your group about why the construction in problem 12 works.

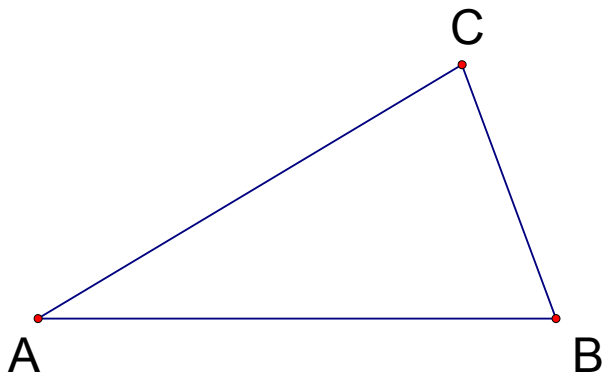
15. Construct an *isosceles* triangle using ST as a *base*.



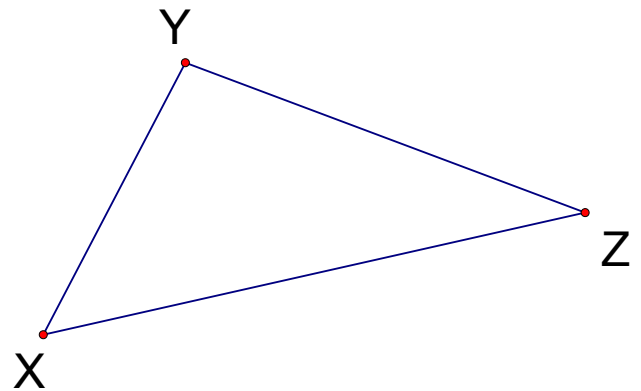
16. Construct an *isosceles* triangle using UV as a *base* and WX as a *leg*.



17. Construct the *perpendicular bisector* of side AB of triangle ABC , and the *median* from vertex C .

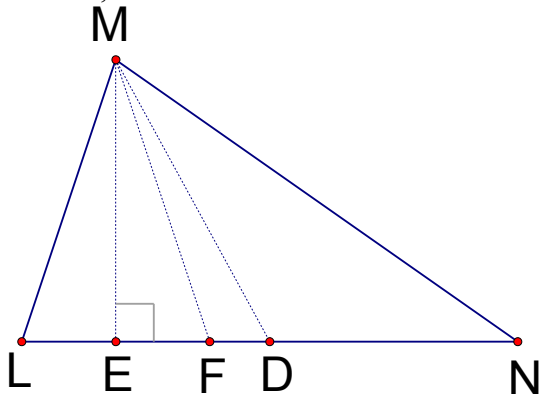


18. Construct the *altitude* from vertex Y in triangle XYZ (hint: see problem 12).

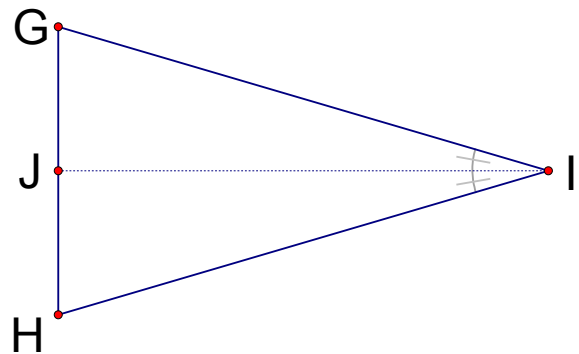


Before we learn the algorithm for bisecting an angle, let's explore the properties of *medians*, *altitudes*, and *angle bisectors* in *scalene* and *isosceles* triangles.

In *scalene* triangle LMN , ME is an *altitude*, MF is an *angle bisector*, and MD is a *median*.



In *isosceles* triangle GHI , IJ is an *angle bisector* of vertex angle I .



Notice in a scalene triangle that the *altitude*, *angle bisector*, and *median* constructed from the same vertex are all different segments. The "feet" of these segments will always be in the same order, with the *angle bisector* between the *altitude* and the *median*.

19. Using isosceles triangle GHI with angle bisector IJ from the bottom of page 4, prove that triangle GJI is congruent to triangle HJI .

20. After proving the triangles are congruent, what can you say about point J ? _____ Why?

21. What can you say about the measures of angles GJI and HJI ? _____ Why?

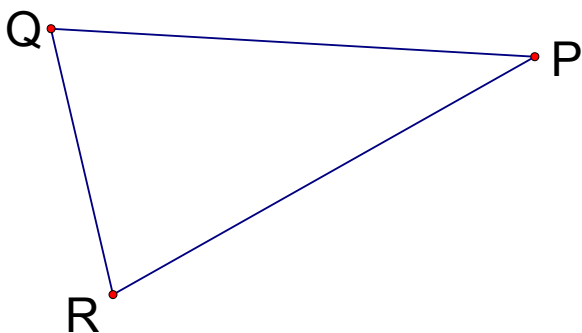
22. Using information from problem number 20, what is another name for segment IJ ? _____

23. Using information from problem number 21, what is a different name for segment IJ ? _____
(with respect to vertex I)

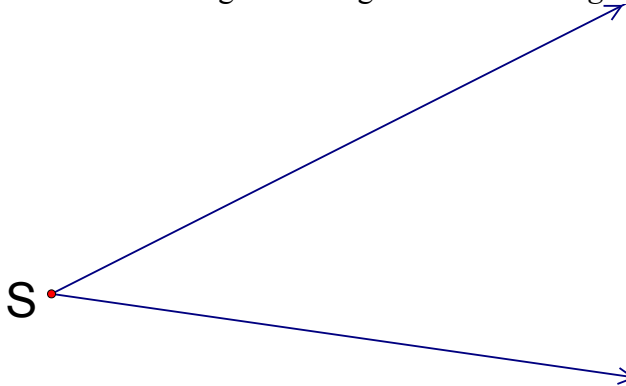
24. Combining the properties of the segments in problems 22 and 23, what is another name for the line containing segment IJ (with respect to side GH)? _____

The preceding properties are summarized in the Isosceles Triangle Vertex Angle Bisector Theorem: In an *isosceles* triangle, the *bisector* of the *vertex angle* is also the *altitude* and *median* to the base (and the perpendicular bisector of the base). In other words, all four segments determine the same line. We can use this fact specifically to construct the angle bisector of the vertex angle in an *isosceles* triangle, and more generally, to bisect any angle.

25. Construct the *angle bisector* of vertex angle P in *isosceles* triangle PQR by constructing the \perp bisector of base QR (you already know one point on the \perp bisector of QR).



26. Construct the *bisector* of angle S (first, construct an *isosceles* triangle with angle S the *vertex angle*)



We now have an *algorithm* for constructing the angle bisector. With your compass at the vertex of the angle, swing an arc that crosses both sides of the angle. The intersection points will be the vertices of the base angles of an isosceles triangle (you don't actually need to draw the base). Construct the \perp bisector of this base by

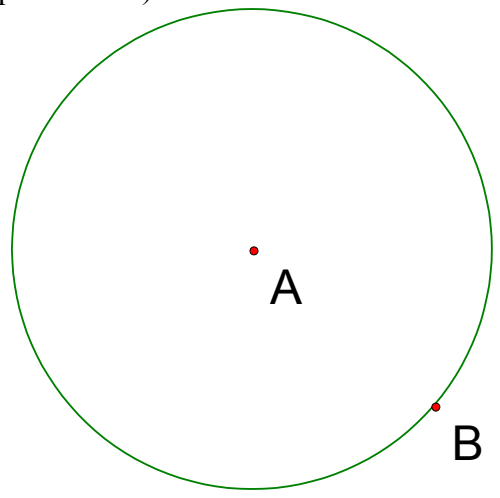
constructing a second point equidistant to the endpoints of the base. The line connecting this point with the vertex will bisect the angle.

An important theorem that we will use when talking about piles of salt is the *Angle Bisector Theorem*, which says that any point on the bisector of an angle is equidistant from the *sides* of the angle. The converse of this theorem says that any point which is equidistant from the sides of an angle is on the angle bisector.

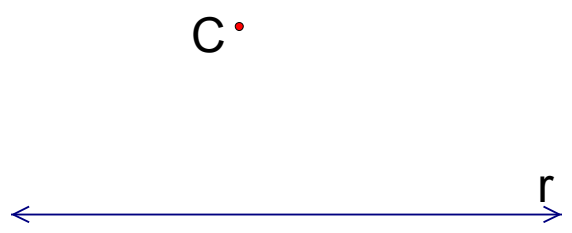
27. Using the Angle Bisector Theorem, write a locus definition for the angle bisector. _____

Important theorem: A *tangent* line to a circle is perpendicular to the radius of the circle at the point of tangency.

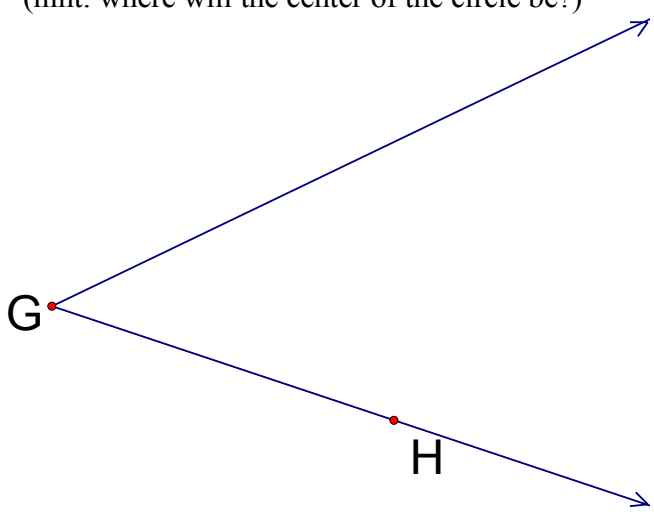
28. Construct the *tangent* to circle A at point B (hint: see problem 11).



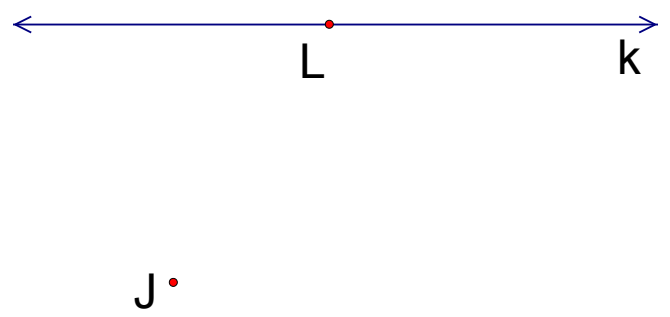
29. Construct the circle with center C and tangent to line r (hint: see problem 12).



30. Construct the circle *inscribed* in angle G (tangent to both sides) that is tangent to side GH at point H . (hint: where will the center of the circle be?)



31. Construct the circle that contains point J and is tangent to line k at point L (hint: problems 16, 30).



Salt Pile Experiment

The *angle of repose* is the minimum angle at which a granular material can no longer support itself, and will flow under the influence of gravity. (The term "granular" covers a wide range, since even large boulders that accumulate at the foot of a mountain have an angle of repose, and a rockslide or avalanche occurs if this angle is exceeded.) The angle of repose is the measure of the angle between the surface of the material and horizontal.

The steepness of the angle of repose is affected by such properties as the size and angularity of the grains, density of the grains, cohesion between the grains (due to electrostatic energy, magnetism, water film, etc.), substrate roughness, shear-stress, and gravity. Ordinary table salt has an angle of repose of about 32° .

Circle Experiment

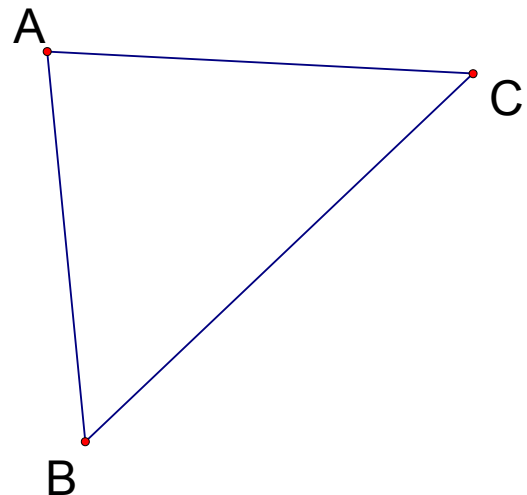
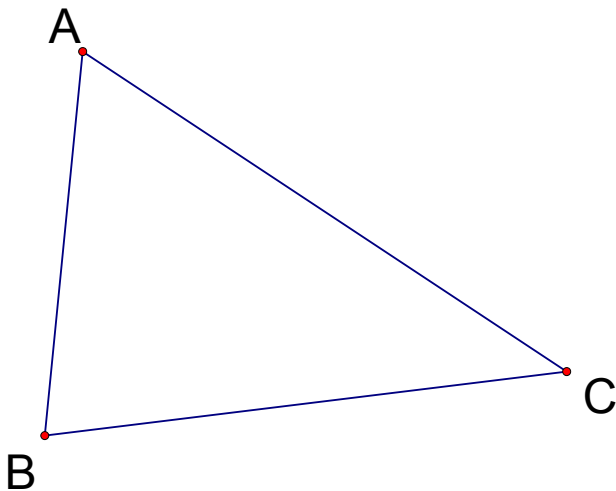
- Predict what will happen when salt is poured on a circular shape.
- Pour salt on circular shape and observe if your predictions are verified.
- Where is the apex of the cone in relation to the circular base?
- If you know the diameter of the circular base, and the angle of repose, can you calculate the height?

Triangle Experiment

- Predict what will happen when salt is poured on a triangular shape.
- Pour salt on triangular shape and observe if your predictions are correct.
- Where is the apex of the pyramid in relation to the triangular base?
- What are the ridgelines in relation to the triangular base?

Construct a model of your observations from the triangular experiment.

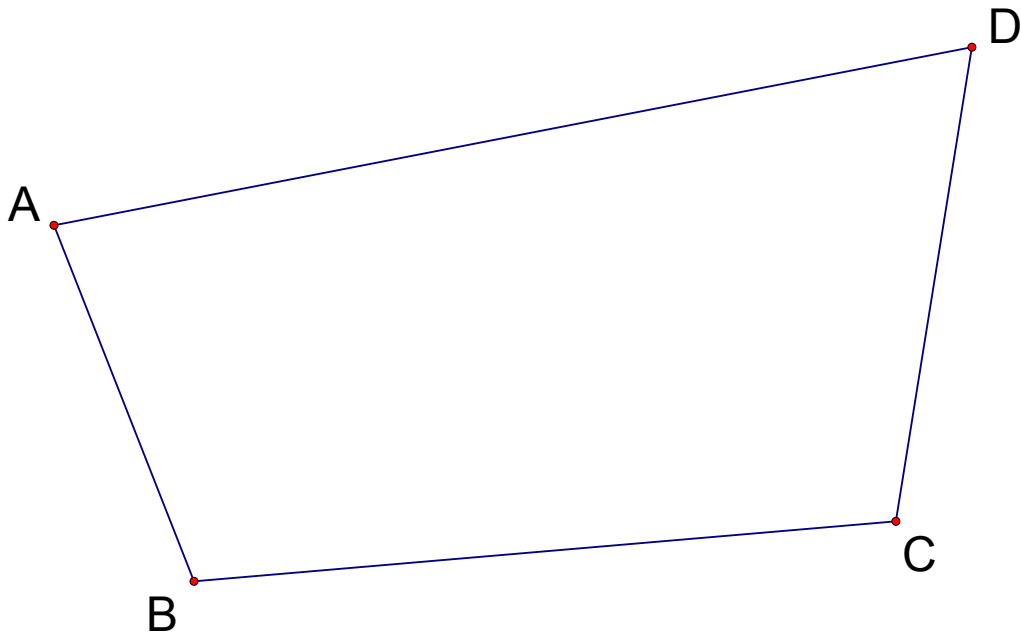
Inscribe a circle in the triangle (Don't forget to construct the point of tangency of the incircle).



Quadrilateral Experiment

- Predict what will happen when salt is poured on a quadrilateral shape.
- Pour salt on quadrilateral shape and observe if your predictions are correct.
- Describe the ridgelines in relation to the quadrilateral base?

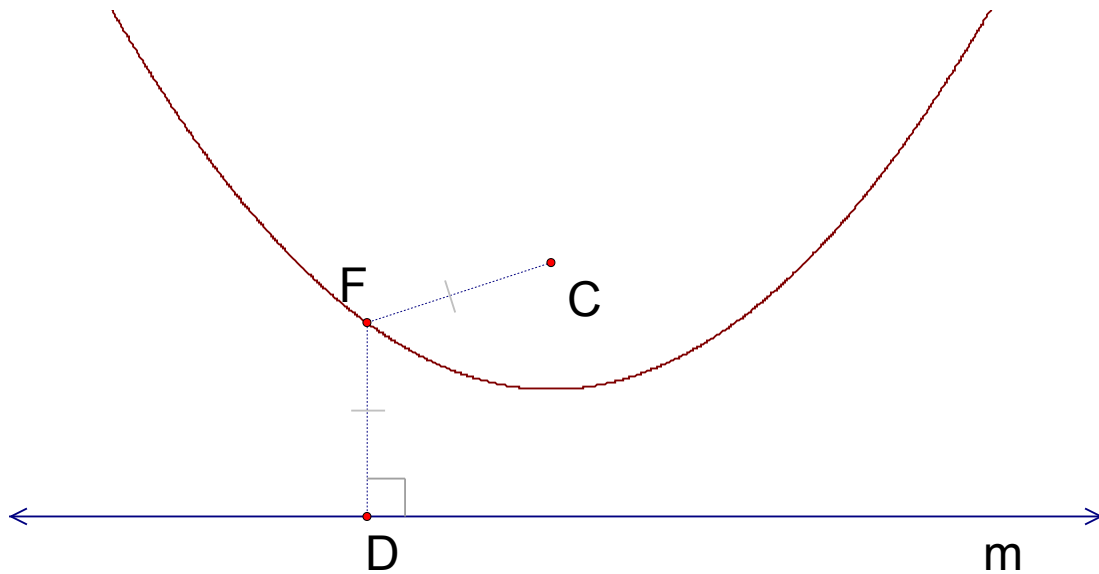
Construct a model of your observations from the triangular experiment.



Circle Near Edge Experiment

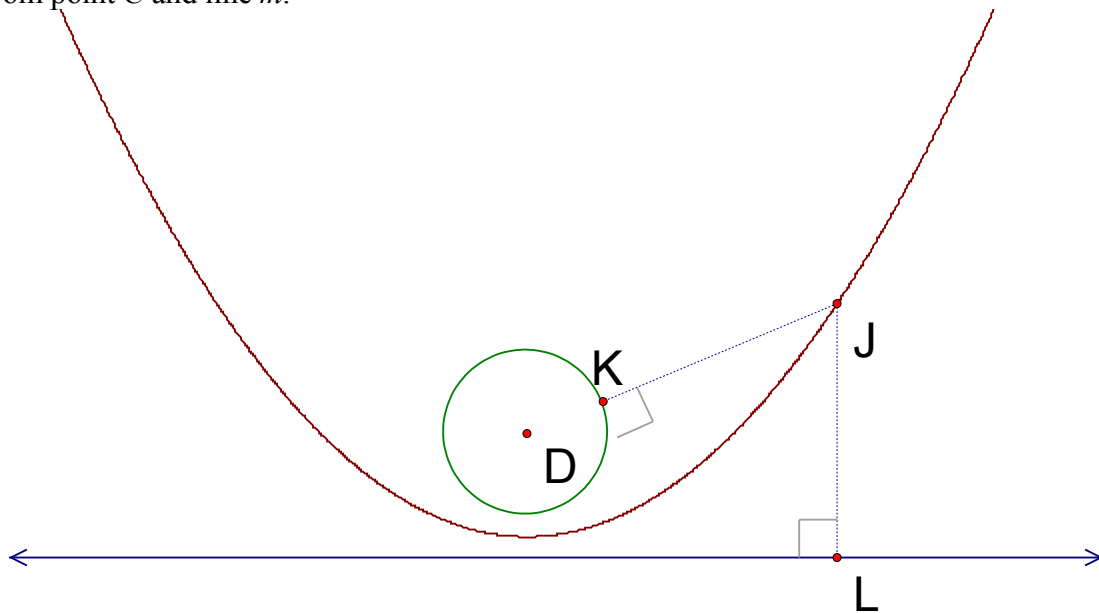
- Predict what will happen when salt is poured on a surface with a hole near the edge.
- Pour salt on surface and observe if your predictions are correct.
- Describe the ridgeline in relation to the edge and the circle.

Definition: A *parabola* is the locus of points equidistant from a *point* (focus) and a *line* (directrix).



The distance from F to line m is measured along the perpendicular from F to line m . This distance FD is the same as the distance FC . Since F is equidistant from points C and D , describe how to construct point F given point D on line m and point C .

Choose another point G on the parabola, and perform the construction that would demonstrate that G is equidistant from point C and line m .



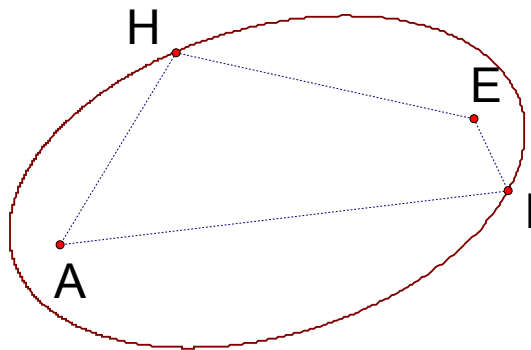
Why is the distance from J to K the same as the distance from J to L ?

The definition of a *parabola* is the locus of points that are equidistant from a *point* and a *line*, not a *circle* and a *line*. Can you prove that the locus is a parabola?

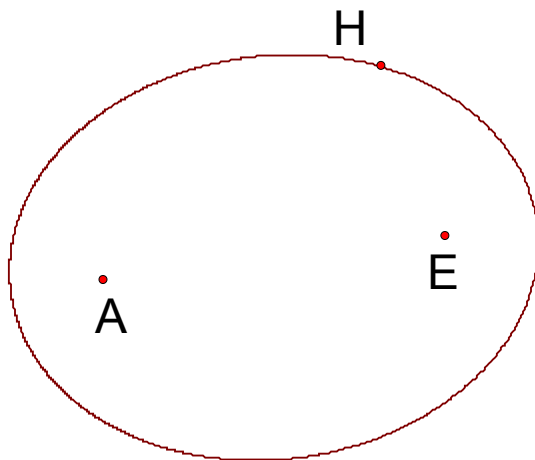
Definition: An *ellipse* is the locus of points whose *sum to two fixed points (foci)* is *constant*.

$$HA + HE = IA + IE = \text{constant}$$

Perform the construction that will demonstrate that H is on the ellipse. (I will guide you)



Practice the construction again.



Why is the distance from J to K the same as the distance from J to E ?

The definition of an *ellipse* is the locus of points whose *sum to two fixed points* is *constant*. Can you prove that the locus is an *ellipse*?

